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THE STABILITY OF SPIRAL FLOW BETWEEN ROTATING CYLINDERS

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CONTENTS

The effect of an axial pressure gradient on the stability of viscous flow between rotating cylinders is discussed on the basis of the narrow gap approximation, the assumption of axisymmetric disturbances, and the assumption that the cylinders rotate in the same direction. The onset of instability then depends on both the Taylor number (T) and the axial Reynolds number (R). For large values of R, the dominant mechanism of instability is of the Tollmien–Schlichting type and the present theory is based therefore on a generalization of the asymptotic methods of analysis that have been developed for the Orr–Sommerfeld equation. The present results, when combined with previous results for small values of R, give the complete stability boundary in the (R, T)-plane. Only limited agreement is found with existing experimental data and it is suggested therefore that it may be necessary to consider either non-axisymmetric disturbances or nonlinear effects.

1. Introduction

In the study of the stability of viscous flows the problem of spiral flow between rotating cylinders is of particular interest because of the interaction it exhibits between the Taylor–Görtler mechanism of instability associated with the rotational flow and the Tollmien–Schlichting mechanism of instability associated with the axial flow. For small values of the Reynolds number the axial flow is known to have a stabilizing effect (Chandrasekhar 1960, 1961, 1962; DiPrima 1960; Krueger & DiPrima 1964; Datta 1965). But this stabilizing effect cannot persist indefinitely, since the flow must ultimately become unstable through the Tollmien–Schlichting mechanism of instability even in the absence of rotation. In the present paper, therefore, we wish to consider the stability of spiral flow for large values of the axial Reynolds number by a generalization of the asymptotic methods of approximation that have been developed for parallel shear flows.

To simplify the problem somewhat we have made three basic approximations. First, we make the well-known narrow gap approximation, i.e. we assume that

$$d = R_2 - R_1 \leqslant \tfrac{1}{2}(R_1 + R_2),$$

which simplifies both the velocity components of the basic flow and the linearized disturbance equations. In this approximation the problem reduces to the narrow gap Taylor problem in the absence of an axial flow and to the problem of plane Poiseuille flow in the absence of rotation. Secondly, we consider only axisymmetric disturbances. This is an assumption that is known to be valid in both limiting cases at least. In the absence of rotation this follows from Squire's theorem (Squire 1933) and in the absence of an axial flow it follows from the results of Krueger, Gross & DiPrima (1966) provided that $\Omega_2/\Omega_1 > -0{\cdot}78$. Thirdly, we assume that the cylinders rotate in the same directions so that one of the coefficients in the governing equations which involves the rotational component of the basic flow can be replaced by its average value. This is also known to be a good approximation in the absence of an axial flow (Chandrasekhar 1961, pp. 309–313) and its validity in the presence of an axial flow has recently been confirmed by Krueger & DiPrima (1964) up to an axial Reynolds number† of 30. This last assumption also has the important consequence of making the governing equations symmetric so that we can then treat the even and odd modes separately.

When $R = 0$ we know that instability sets in at a critical Taylor number‡ of 1708 and leads to a steady secondary flow in the form of Taylor vortices. For small values of R, formal perturbation theories have also been developed by Chandrasekhar (1962) and Datta (1965), both of which lead to a result of the form

$$T_c(R) = T_0 + T_2 R^2 + \dots, \tag{1·1}$$

where $T_0 = 1708$ and T_2 is a positive constant. The value of T_2 originally given by Chandrasekhar was shown to be in error by Krueger & DiPrima (1964) and was subsequently corrected by Datta, who found that $T_2 = 2{\cdot}35$ (based on his own theory) and $T_2 = 2{\cdot}8$ (based on Chandrasekhar's theory). Both theories are sufficiently complicated, however, that the discrepancy between these values cannot easily be resolved. The Galerkin method has also been used by DiPrima (1960) and Krueger & DiPrima (1964) to compute T_c for R in the range $0 \leqslant R \leqslant 45$.

For larger values of R, the asymptotic methods of analysis that have been developed for parallel shear flows are generally applicable, though they must be generalized in some important respects. In this approach it is convenient to replace the pair of parameters R and T,

† The Reynolds number R used in the present paper is based on $\frac{1}{2}d$ and the maximum velocity of the axial flow, whereas the Reynolds number $\mathcal{R}$ used by Chandrasekhar (1961, 1962) and some other writers is based on d and the mean velocity. They are related by $R = \frac{3}{4}\mathcal{R}$.

‡ The 'modified' Taylor number T used in the present paper is based on a normalization that is appropriate for the case in which the cylinders rotate in the same direction. Chandrasekhar (1961) denotes it by $\bar{T}$.

which both depend on viscosity, by an equivalent pair R and β, where $\beta = \frac{1}{4}T^{\frac{1}{2}}/R$ is independent of viscosity. The asymptotic approximations developed in the present paper are therefore based on letting $R \to \infty$ for fixed values of β.

2. The governing equations

Let the components of the basic flow in polar coordinates be given by $(0, V, W)$, where $V(r)$ is the azimuthal component of the velocity due to the rotation of the cylinders and $W(r)$ is the axial component of the velocity due to the axial pressure gradient. If we denote the components of the disturbance velocity by (u_r, u_θ, u_z), then the linearized equations for axisymmetric disturbances are (cf. Chandrasekhar 1961, p. 372)

$$\frac{\partial u_r}{\partial t}+W\frac{\partial u_r}{\partial z}-2\Omega u_\theta = -\frac{\partial \varpi}{\partial r}+\nu\left(\nabla^2-\frac{1}{r^2}\right)u_r, \tag{2·1}$$

$$\frac{\partial u_\theta}{\partial t}+W\frac{\partial u_\theta}{\partial z}+\left(\frac{\mathrm{d}V}{\mathrm{d}r}+\frac{V}{r}\right)u_r = \nu\left(\nabla^2-\frac{1}{r^2}\right)u_\theta, \tag{2·2}$$

$$\frac{\partial u_z}{\partial t}+W\frac{\partial u_z}{\partial z}+\frac{\mathrm{d}W}{\mathrm{d}r}u_r = -\frac{\partial \varpi}{\partial z}+\nu\nabla^2 u_z \tag{2·3}$$

and

$$\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z} = 0, \tag{2·4}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}. \tag{2·5}$$

In these equations we have let $\Omega = V/r$ and $\varpi = \delta p/\rho$.

It is convenient at this point to introduce a stream function ψ to describe the flow in planes which pass through the axis. Thus, if we let

$$u_r = -\frac{\partial \psi}{\partial z} \quad \text{and} \quad u_z = +\frac{\partial \psi}{\partial r}+\frac{\psi}{r} \tag{2·6}$$

then the equation of continuity (2·4) is automatically satisfied. On eliminating ϖ between equations (2·1) and (2·3) we then have

$$\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)\left(\nabla^2-\frac{1}{r^2}\right)\psi-\Psi(r)\frac{\partial \psi}{\partial z}+2\Omega\frac{\partial u_\theta}{\partial z} = \nu\left(\nabla^2-\frac{1}{r^2}\right)^2\psi, \tag{2·7}$$

where

$$\Psi(r) = \frac{\mathrm{d}^2W}{\mathrm{d}r^2}-\frac{1}{r}\frac{\mathrm{d}W}{\mathrm{d}r}. \tag{2·8}$$

Equation (2·2) can also be written in the form

$$\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)u_\theta-(\mathrm{D}_* V)\frac{\partial \psi}{\partial z} = \nu\left(\nabla^2-\frac{1}{r^2}\right)u_\theta, \tag{2·9}$$

where

$$\mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}r} \quad \text{and} \quad \mathrm{D}_* = \mathrm{D}+\frac{1}{r}. \tag{2·10}$$

In a normal mode analysis of these equations there are two somewhat different ways of proceeding. We can take a form for the normal modes such that the governing equations reduce, when $W = 0$, to Taylor's equation or, when $V = 0$ (and we make the narrow gap approximation), to the Orr–Sommerfeld equation. In past work on this problem it is mainly the first of these possibilities that has been considered. Here, however, we wish to consider the second of these possibilities and for this purpose we let

$$\psi(r,z,t) = \phi(r)\,\mathrm{e}^{\mathrm{i}\alpha(z-ct)} \quad \text{and} \quad u_\theta(r,z,t) = v(r)\,\mathrm{e}^{\mathrm{i}\alpha(z-ct)}. \tag{2·11}$$

Equations (2·7) and (2·9) then become

$$\nu(\mathrm{DD}_* - \alpha^2)^2\phi = \mathrm{i}\alpha(W-c)\,(\mathrm{DD}_*-\alpha^2)\,\phi - \mathrm{i}\alpha\Psi(r)\,\phi + 2\mathrm{i}\alpha\Omega v \tag{2·12}$$

and

$$\nu(\mathrm{DD}_*-\alpha^2)\,v = \mathrm{i}\alpha(W-c)\,v - \mathrm{i}\alpha(\mathrm{D}_*V)\,\phi. \tag{2·13}$$

Consider now the reduction of equations (2·12) and (2·13) to non-dimensional form within the framework of the narrow gap approximation. Since we want the resulting equations to reduce (when $V = 0$) to the Orr–Sommerfeld equation with the usual notation for plane Poiseuille flow, we choose the characteristic length $L_* = \frac{1}{2}d = \frac{1}{2}(R_2 - R_1)$ and then introduce the non-dimensional radial coordinate

$$y = \frac{r}{L_*} - \frac{R_1+R_2}{2L_*} \tag{2·14}$$

so that y lies in the usual interval $-1 \leqslant y \leqslant +1$. In the narrow gap approximation we have

$$W(r) \cong W_*(1-y^2), \tag{2·15}$$

where W_* denotes the maximum value of the axial flow which we also choose for our characteristic velocity. In the same approximation we also have

$$V(r) \cong R_1\Omega_1\omega(y) \quad \text{and} \quad \Omega(r) \cong \Omega_1\omega(y), \tag{2·16}$$

where

$$\omega(y) = \tfrac{1}{2}[(1+\mu)-(1-\mu)\,y] \quad \text{and} \quad \mu = \Omega_2/\Omega_1. \tag{2·17}$$

Before we write down the narrow gap forms of equations (2·12) and (2·13) it is necessary to choose suitable non-dimensional forms for ϕ and v. A convenient choice for this purpose would appear to be

$$\phi = L_*W_*\phi_* \quad \text{and} \quad v = \tfrac{1}{2}(1-\mu)\,R_1\Omega_1 v_*; \tag{2·18}$$

we also let

$$\alpha = \alpha_*/L_* \quad \text{and} \quad c = c_*W_*. \tag{2·19}$$

On making these substitutions in equations (2·12) and (2·13), taking the narrow gap limit, and then dropping the asterisks, we have

$$(\mathrm{i}\alpha R)^{-1}\,(\mathrm{D}^2-\alpha^2)^2\phi = (U-c)\,(\mathrm{D}^2-\alpha^2)\,\phi - U''\phi + \beta^2\frac{\omega(y)}{\frac{1}{2}(1+\mu)}\,v \tag{2·20}$$

and

$$(\mathrm{i}\alpha R)^{-1}\,(\mathrm{D}^2-\alpha^2)\,v = (U-c)\,v + \phi, \tag{2·21}$$

where

$$\left.\begin{aligned} \mathrm{D} &= \frac{\mathrm{d}}{\mathrm{d}y}, \quad U = 1-y^2, \quad R = \frac{W_*L_*}{\nu}, \\ \text{and} \quad \beta^2 &= \tfrac{1}{4}(1-\mu)\,(1+\mu)\left(\frac{R_1\Omega_1}{W_*}\right)^2\frac{d}{R_1}. \end{aligned}\right\} \tag{2·22}$$

The presence of $\omega(y)$ in equation (2·20) introduces an asymmetry into that equation which would lead to severe difficulties in the subsequent asymptotic treatment of the problem. To avoid such difficulties we now make the further approximation of replacing $\omega(y)$ by its average value $\langle\omega(y)\rangle = \frac{1}{2}(1+\mu)$. In the absence of an axial flow this is known to be a good approximation (Chandrasekhar 1961, pp. 309–313); in the presence of an axial flow its validity has been confirmed by Krueger & DiPrima (1964) for values of R up to 30. Thus, our governing equations become

$$(\mathrm{i}\alpha R)^{-1}(\mathrm{D}^2-\alpha^2)^2\phi = (U-c)(\mathrm{D}^2-\alpha^2)\phi - U''\phi + \beta^2 v \tag{2·23}$$

and

$$(\mathrm{i}\alpha R)^{-1}(\mathrm{D}^2-\alpha^2)v = (U-c)v + \phi \tag{2·24}$$

together with the boundary conditions

$$\phi = \mathrm{D}\phi = v = 0 \quad \text{at} \quad y = \pm 1. \tag{2·25}$$

These equations are now symmetrical in y and we can therefore consider the even and odd solutions separately. When $\beta = 0$, equation (2·23) reduces to the Orr–Sommerfeld equation and in that limit instability is known to be associated with an even solution. Also, when $\beta \to \infty$, a simple renormalization of equations (2·23) and (2·24) leads to Taylor's equation and in that limit the onset of instability is also known to be associated with an even solution. Accordingly we shall consider the even solution of equations (2·23) and (2·24) which tends to the appropriate limits as $\beta \to 0$ and $\beta \to \infty$.

The coupling parameter β which appears in equation (2·23) plays a crucial role throughout the entire analysis. As we have defined it, β is independent of the viscosity and in our asymptotic treatment of the problem we shall proceed on the hypothesis that β is fixed and $R \to \infty$. To compare our results with previous ones, however, it is convenient to note here that

$$\beta = \tfrac{1}{4}T^{\frac{1}{2}}/R \tag{2·26}$$

where T is the modified Taylor number

$$T = -2A\Omega_1(1+\mu)\,d^4/\nu^2 \tag{2·27}$$

and $A = \Omega_1(\mu R_2^2 - R_1^2)/(R_2^2 - R_1^2)$. In the narrow gap approximation this can be written in the form

$$T \simeq (1-\mu)(1+\mu)\left(\frac{\Omega_1 R_1 d}{\nu}\right)^2 \frac{d}{R_1}. \tag{2·28}$$

The results obtained by Krueger & DiPrima (1964) based on equations equivalent to equations (2·23) and (2·24), i.e. based on the approximation of replacing $\omega(y)$ by its average value, cover the range $0 \leqslant R \leqslant 45$; in terms of β this corresponds to $0{\cdot}5 \leqslant \beta \leqslant \infty$. Accordingly we shall be primarily concerned with small values of β lying in the range $0 \leqslant \beta \leqslant 0{\cdot}5$.

Before we proceed with the detailed analysis of the problem it is convenient to rewrite the governing equations in a somewhat more compact form. For this purpose let

$$L_2 \equiv (\mathrm{i}\alpha R)^{-1}(\mathrm{D}^2-\alpha^2) - (U-c) \tag{2·29}$$

and

$$L_4 \equiv (\mathrm{i}\alpha R)^{-1}(\mathrm{D}^2-\alpha^2)^2 - (U-c)(\mathrm{D}^2-\alpha^2) + U''. \tag{2·30}$$

The governing equations (2·23) and (2·24) can then be written in the simple form

$$L_4\phi = \beta^2 v \quad \text{and} \quad L_2 v = \phi \tag{2·31}$$

from which we choose to eliminate v to obtain

$$L_2 L_4 \phi = \beta^2\phi. \tag{2·32}$$

For an even solution we impose the boundary conditions

$$\phi = \phi' = L_4\phi = 0 \quad \text{at} \quad y = -1 \quad \text{and} \quad \phi' = \phi''' = \phi^{\mathrm{v}} = 0 \quad \text{at} \quad y = 0. \tag{2·33}$$

We thus have a characteristic value problem and in the following sections we shall derive a number of different asymptotic approximations to the solutions of equation (2·32), together with the corresponding characteristic equations which then follow from the boundary conditions (2·33).

3. The inviscid solutions

In attempting to construct approximations to the solutions of equation (2·32) by the heuristic asymptotic methods that have been developed for the Orr–Sommerfeld equation (see, for example, Lin 1955 or Reid 1965), a natural starting point is to consider a formal expansion in inverse powers of $\mathrm{i}\alpha R$ of the form

$$\phi(y) = \phi^{(0)}(y) + (\mathrm{i}\alpha R)^{-1}\,\phi^{(1)}(y) + \ldots, \tag{3·1}$$

where the first approximation $\phi^{(0)}(y)$ satisfies the inviscid equation

$$(U-c)\,(\mathrm{D}^2-\alpha^2)\,\phi - U''\phi - \frac{\beta^2}{U-c}\phi = 0. \tag{3·2}$$

An equation of this type is familiar from the inviscid stability theory for parallel shear flows in a stratified fluid (see, for example, Drazin & Howard 1966). In that context, however, $-\beta^2$ is replaced by a Richardson number which is then usually taken to be positive corresponding to a stably stratified fluid. Furthermore, our interest in equation (3·2) is somewhat different, the basic question being to what extent the solutions of equation (3·2) provide approximations to the solutions of the full equation (2·32).

A point $y = y_c$ where $U-c$ vanishes but $U'(y_c) \neq 0$ is a regular singular point of equation (3·2) with exponents p_1 and p_2, where p_1 and p_2 are the roots of the indicial equation

$$p(p-1) - (\beta/U_c')^2 = 0, \tag{3·3}$$

i.e.

$$p_1, p_2 = \tfrac{1}{2}\{1 \pm [1+(2\beta/U_c')^2]^{\frac{1}{2}}\}. \tag{3·4}$$

Note that these roots satisfy the simple relation $p_1+p_2 = 1$. Thus, provided these roots do not differ by an integer (i.e. $\beta/U_c' \neq 0, \tfrac{1}{2}\sqrt{3}, \sqrt{2}, \ldots$), we have two solutions of the form

$$\phi_1(y,\beta) = (y-y_c)^{p_1} P_1(y-y_c) \quad \text{and} \quad \phi_2(y,\beta) = (y-y_c)^{p_2} P_2(y-y_c), \tag{3·5}$$

where $P_1(y-y_c)$ and $P_2(y-y_c)$ are power series in $y-y_c$ with leading terms of unity. Since both of these solutions have algebraic branch points at the critical point, neither of them can provide uniformly valid asymptotic approximations to any solution of equation (2·32) in a full complex neighbourhood of y_c. By considering the 'viscous corrections' to ϕ_1 and ϕ_2, however, it can be shown (see § 9 and the Appendix) that they do provide valid asymptotic approximations in the usual sector $-\tfrac{7}{6}\pi < \arg(y-y_c) < \tfrac{1}{6}\pi$ of the complex y-plane.

When the roots of the indicial equation differ by an integer, the solution (3·5) for ϕ_2 is not satisfactory and other forms must be found. Since β/U_c' never becomes as large as $\tfrac{1}{2}\sqrt{3}$ in the present calculations, we need only consider the limiting case $\beta = 0$. In that case, as is well known, ϕ_2 must be of the form

$$\phi_2(y) = P_2(y-y_c) + (U_c''/U_c')\,\phi_1(y)\ln(y-y_c), \tag{3·6}$$

where, as usual, we suppose that ϕ_2 contains no multiple of ϕ_1, i.e. that the coefficient of $y-y_c$ in $P_2(y-y_c)$ is zero.

In the computation of these inviscid solutions there are two essentially different methods that can be used. One is based on the direct summation of the power series representations

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of the $P_i(y-y_c)$, $i = 1, 2$. The coefficients of the powers of $y-y_c$ satisfy simple five-term recursion formulae and the summation of these series on a digital computer is therefore not difficult. If the summation proceeds until the absolute value of the ratio of the nth term to the nth partial sum is less than ϵ (usually $\epsilon = 5\times 10^{-9}$), then in all cases of interest fewer than forty terms of the series are needed. To insure that an unstable situation was not present owing to round-off errors in the repeated use of the recursion formulae, sample summations were repeated using double precision arithmetic and were found to agree within acceptable limits. Since the radius of convergence of the series is $2|y_c|$, the summation procedure can be used to compute the values of the P_i and their derivatives at $y = -1$ and $y = 0$ for $-1 < y_c < -\frac{1}{3}$, i.e. $0 < c < \frac{8}{9}$.

The second method of computing the inviscid solutions is based on the numerical integration of the differential equations satisfied by the $P_i(y-y_c)$. They are then defined as the regular solutions of the differential equations

$$P_i'' + \frac{2p_i}{y-y_c}P_i' - \alpha^2 P_i - \frac{U''}{U-c}P_i - \beta^2\left\{\frac{1}{(U-c)^2} - \frac{1}{U_c'^2(y-y_c)^2}\right\}P_i = 0 \tag{3·7}$$

that satisfy the initial conditions

$$P_i(0) = 1 \quad \text{and} \quad P_i'(0) = \left(1 - p_i + \frac{1}{p_i}\right)\frac{U_c''}{2U_c'}. \tag{3·8}$$

A fourth-order single precision Runge–Kutta method was used to integrate these equations. In order to obtain the required starting values, the power series for the P_i were first evaluated at the points $y_c^{\pm}$ (say), where $y_c^+ > y_c$ and $y_c^- < y_c$. The interval $[-1, 0]$ was then divided into eighty equal segments and the $y_c^{\pm}$ were chosen to be the division points nearest to y_c such that $|y_c^{\pm}| - |y_c| > 0·05$. The differential equations (3·7) were then integrated from y_c^+ to 0 and from y_c^- to -1.

Since we shall consider only the characteristic values of the problem, we need only obtain the values of $P_i(y-y_c)$ and $P_i'(y-y_c)$ at the boundary points $y = -1$ and $y = 0$. In this case the summation of the power series was found to be more than five times faster than the numerical integration of the initial value problem.

4. The viscous solutions of W.K.B. type

In the derivation of approximations to the solutions of equation (2·32) of viscous type there are a number of different approaches that can be used. In this section we shall consider the solutions of W.K.B. type which provide approximations that are valid in certain domains of the complex y-plane outside the neighbourhood of the point y_c. One of the boundary points ($y = -1$), however, lies in a domain in which they are not valid (at least not in the complete sense of Olver†). To obtain approximations that are valid at this boundary

† The concept of a 'complete' asymptotic expansion has been developed by Olver (1961, 1963, 1964) in connexion with his theory of error bounds for asymptotic solutions of certain second-order differential equations. But the application of this concept in the present paper goes considerably beyond what has been rigorously established by him. The essential idea, however, is the importance of restricting the use of different asymptotic expansions of a given function to non-overlapping domains (the boundaries of which are Stokes lines) even though the expansions may remain valid in the usual Poincaré sense in larger overlapping domains. Throughout this paper, therefore, we shall usually specify the domains of validity in this more restrictive sense.

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point it will be necessary to consider approximations of the local turning point type which can then be used to extend the domain of validity of the W.K.B. solutions. By comparing these two types of approximations it is also possible to construct, in a heuristic way, composite approximations of the Tollmien type.

The W.K.B. solutions of equation (2·32) have been briefly discussed previously by Koppel (1964) in a somewhat different context. He considered the stability of a thermally stratified fluid in a parallel shear flow for which the governing equation is similar to equation (2·32) but with an additional dependence on the Prandtl number, and he found that the W.K.B. solutions have different forms depending on whether the Prandtl number is equal to unity or not. When the stratification is unstable and the Prandtl number is one there is an exact mathematical analogy between the two problems and in that case our results are in complete agreement with Koppel's so far as they overlap. Our results go somewhat beyond Koppel's, however, particularly in the discussion of the slowly varying parts of the solutions, in the subsequent matching of the W.K.B. solutions to the local turning point solutions, and in the construction of composite approximations.

To derive the solutions of W.K.B. type, let

$$\phi = \exp\left\{\int g\,\mathrm{d}y\right\}, \tag{4·1}$$

so that g satisfies the non-linear equation

$$\begin{aligned}(\mathrm{i}\alpha R)^{-2}\{g^6+15g^4g'+20g^3g''+45g^2g'^2-3\alpha^2g^4+\ldots\}&\\ -(\mathrm{i}\alpha R)^{-1}\{2(U-c)\,(g^4+6g^2g'+4gg''+3g'^2-2\alpha^2g^2+\ldots)+2U'(g^3+3gg'+\ldots)+\ldots\}&\\ +(U-c)^2\,(g^2+g'-\alpha^2)-(U-c)\,U''=\beta^2,&\end{aligned} \tag{4·2}$$

where only those terms have been written down which will be needed in the subsequent analysis. This equation is then solved in the usual way by assuming an expansion for g of the form

$$g(y) = (\mathrm{i}\alpha R)^{\frac{1}{2}}\,g_0(y)+g_1(y)+(\mathrm{i}\alpha R)^{-\frac{1}{2}}\,g_2(y)+\ldots. \tag{4·3}$$

On formally substituting this expansion into equation (4·2) and equating to zero the coefficients of like powers of $(\mathrm{i}\alpha R)^{\frac{1}{2}}$, we obtain a sequence of equations for the determination of $g_0, g_1, \ldots$. The first equation in this sequence yields simply

$$g_0^2\{g_0^2-(U-c)\}^2 = 0, \tag{4·4}$$

so that

$$\textit{either}\quad g_0^2 = 0 \quad \textit{or}\quad g_0^2 = U-c. \tag{4·5}$$

The next equation, however, which normally determines g_1, is found to vanish identically for both of the above values of g_0^2. Thus, g_1 is not determined to this order. Considering next the third equation of the sequence, we find that the coefficient of g_2 in this equation automatically vanishes, as indeed it must for consistency. By letting $g_0^2 = 0$ in this equation we have

$$(U-c)^2\,(g_1'+g_1^2-\alpha^2)-(U-c)\,U'' = \beta^2, \tag{4·6}$$

which is a first-order non-linear equation equivalent to the inviscid equation (3·2). Thus, corresponding to the first of the roots (4·5), we simply recover the usual inviscid solutions. On letting $g_0^2 = U-c$, however, we find that the coefficient of α^2 vanishes and we then have

$$(U-c)^2\,(g_1'+g_1^2)+3(U-c)\,U'g_1+\tfrac{5}{4}(U-c)\,U''+\tfrac{15}{16}U'^2 = \tfrac{1}{4}\beta^2. \tag{4·7}$$

And this is the required equation for the determination of the slowly varying parts of the W.K.B. solutions.

In our discussion of the solutions of equation (4·7) it is convenient to transform it back into a second-order linear equation. For this purpose we let

$$g_1 = G'/G, \tag{4·8}$$

so that G satisfies

$$(U-c)^2 G'' + 3(U-c)\, U'G' + \{\tfrac{5}{4}(U-c)\, U'' + \tfrac{15}{16} U'^2 - \tfrac{1}{4}\beta^2\} G = 0. \tag{4·9}$$

The critical point y_c is a regular singular point of this equation with exponents

$$q_1 = -\tfrac{1}{2}p_1 - \tfrac{3}{4} \quad \text{and} \quad q_2 = -\tfrac{1}{2}p_2 - \tfrac{3}{4}, \tag{4·10}$$

where p_1 and p_2 are the roots of the indicial equation (3·3). Thus provided these exponents do not differ by an integer (i.e. $\beta/U_c' \neq \frac{1}{2}\surd 3, \frac{1}{2}\surd 15, \ldots$), we have two solutions of the form

$$G_1(y) = (y-y_c)^{q_1}\, Q_1(y-y_c) \quad \text{and} \quad G_2(y) = (y-y_c)^{q_2}\, Q_2(y-y_c), \tag{4·11}$$

where $Q_1(y-y_c)$ and $Q_2(y-y_c)$ are power series in $y-y_c$ with leading terms of unity.

When the exponents do differ by an integer, $G_2(y)$ remains valid but $G_1(y)$ must be of the form

$$G_1(y) = (y-y_c)^{q_1}\, Q_1(y-y_c) + CG_2(y) \ln (y-y_c), \tag{4·12}$$

where C must be determined so that $Q_1(y-y_c)$ is a regular power series in $y-y_c$ with a leading term of unity. To make this second solution definite we can also require, for example, that it contain no multiple of $G_2(y)$. The presence of this logarithmic term in $G_1(y)$, however, does not affect the later matching of the W.K.B. solutions to the local turning point solutions or the combining of them to yield composite approximations of the Tollmien type. In the calculations that we have done the value of β/U_c' never becomes as large as $\frac{1}{2}\surd 3$ and it is unnecessary, therefore, to consider these special cases further.

Combining these results we then have four W.K.B. solutions of the form

$$\phi_{3,4}(y) = \tfrac{1}{2}\pi^{-\frac{1}{2}}\epsilon^{-q_1} G_1(y) \exp\{\mp (\alpha R)^{\frac{1}{2}} Q(y)\} \tag{4·13}$$

and

$$\phi_{5,6}(y) = \tfrac{1}{2}\pi^{-\frac{1}{2}}\epsilon^{-q_2} G_2(y) \exp\{\mp (\alpha R)^{\frac{1}{2}}\, Q(y)\}, \tag{4·14}$$

where

$$Q(y) = \int_{y_c}^{y} \{\mathrm{i}(U-c)\}^{\frac{1}{2}}\, \mathrm{d}y \tag{4·15}$$

and the normalization has been chosen to facilitate comparison with limiting forms of other approximations. We shall choose the branch of $\{\mathrm{i}(U-c)\}^{\frac{1}{2}}$ in equation (4·15) so that

$$\mathscr{R}\{Q(y)\} > 0 \quad \text{for} \quad \arg(y-y_c) = 0 \quad \text{and} \quad y-y_c > 0.$$

The solutions ϕ_4 and ϕ_6 are then dominant and, to satisfy the boundary conditions of the problem, they must be rejected. The remaining solutions, ϕ_3 and ϕ_5, are then subdominant in the domain bounded by the adjacent anti-Stokes lines and the domain in which they are valid (in Olver's sense) is bounded by the Stokes lines which, near y_c, are given by

$$\arg(y-y_c) = -\tfrac{5}{6}\pi \quad \text{and} \quad \tfrac{1}{2}\pi.$$

In this strict sense, however, they are not valid in the domain which contains the boundary point $y = -1$.

5. The viscous solutions of the local turning point type

The local turning point approximations which will be developed in this section are a natural generalization of the approximations commonly used in stability calculations for parallel shear flows. When used to derive the characteristic equation they provide adequate approximations so long as the resulting values of c are not too large. As was mentioned in the previous section, however, solutions of this type can also be used not only to obtain the connexion formulas for the W.K.B. solutions but also, when suitably combined with the W.K.B. solutions, to provide composite approximations of the Tollmien type.

To derive approximations of this type we first make the transformation

$$\phi(y) = \chi(\xi), \quad \text{where} \quad \xi = (y-y_c)/\epsilon \quad \text{and} \quad \epsilon = (\mathrm{i}\alpha R U_c')^{-\frac{1}{3}}, \tag{5·1}$$

and then expand the solution in powers of ϵ in the form

$$\chi(\xi, \epsilon) = \chi^{(0)}(\xi) + \epsilon\chi^{(1)}(\xi) + \ldots. \tag{5·2}$$

The first approximation $\chi^{(0)}$ (ξ) satisfies the equation

$$\mathrm{A}^2\mathrm{D}^2\chi = (\beta/U_c')^2\chi, \tag{5·3}$$

where
$$\mathrm{D} = \mathrm{d}/\mathrm{d}\xi \quad \text{and} \quad \mathrm{A} = \mathrm{D}^2 - \xi. \tag{5·4}$$

Equation (5·3) clearly plays the same role in the present theory as the equation $\mathrm{AD}^2\chi = 0$ does in the usual theory of the Orr–Sommerfeld equation. It is necessarily of the sixth order; for, among its solutions, we must be able to find two that will provide the required viscous corrections to the leading terms of the singular inviscid solutions and thereby determine the domain of validity of the inviscid solutions. From these remarks it is clear that the term involving β/U_c' in equation (5·3) is of crucial importance.

The fact that ξ appears quadratically in equation (5·3) would suggest, at first sight, that we have reached an impasse. Fortunately, however, this is not the case, for we will now show that all of the solutions of equation (5·3) can be obtained from the two third-order equations

$$(\mathrm{AD} + p_i)\,\chi = 0, \tag{5·5}$$

where p_i $(i = 1, 2)$ are the roots (3·4) of the indicial equation (3·3). To prove this result we first note the identity

$$\mathrm{AD} = \mathrm{DA} + 1. \tag{5·6}$$

Consider next the product $(\mathrm{AD}+p)$ $(\mathrm{AD}-p+1)$, the factors of which obviously permute. On using the identity (5·6) we then have

$$(\mathrm{AD}+p)\,(\mathrm{AD}-p+1)\,\chi = \{\mathrm{A}^2\mathrm{D}^2 - p(p-1)\}\chi, \tag{5·7}$$

which is just (5·3) provided $p(p-1) = (\beta/U_c')^2$, i.e. provided p is a root of the indicial equation (3·3). Thus, provided $p \neq \frac{1}{2}$, i.e. $p_1 \neq p_2$ (and this value is excluded in the present problem since $p_1 \geqslant 1$ and $p_2 \leqslant 0$), the solutions of the sixth-order comparison equation (5·3) can all be obtained from the two third-order equations (5·5). This factorization of equation (5·3) is one of the essential steps in the present theory and results in an important simplification of the problem.†

† In the closely related problem of thermal instability in the presence of a parallel shear flow considered by Koppel (1964) this factorization is not possible if the Prandtl number is different from unity. Although Koppel showed that integral representations of the solutions can still be obtained, they are of a more complicated form with kernels that involve Whittaker functions and the subsequent analysis then becomes much more difficult.

An equation of the type (5·5) has appeared previously in the work of Langer (1955), Hershenov (1957), and Rabenstein (1958) in connexion with the construction of uniform asymptotic approximations to the solutions of the Orr–Sommerfeld equation. Their results are not easily adapted for the present purposes and in the appendix, therefore, we have defined certain standard solutions of equation (5·5) and have discussed some of their properties.

The solutions of equation (5·3) can be conveniently indexed by letting the solutions of equation (5·5) corresponding to the roots p_1 and p_2 be denoted by (χ_1, χ_3, χ_4) and (χ_2, χ_5, χ_6) respectively. From the results given in the appendix we then have

$$\left.\begin{aligned} \chi_1(\xi) &= \epsilon^{p_1} B_3(\xi, p_1), & \chi_2(\xi) &= \epsilon^{p_2} Q_3(\xi, p_2), \\ \chi_3(\xi) &= A_1(\xi, p_1), & \chi_5(\xi) &= A_1(\xi, p_2), \\ \chi_4(\xi) &= A_2(\xi, p_1), & \chi_6(\xi) &= A_2(\xi, p_2). \end{aligned}\right\} \tag{5·8}$$

The solutions χ_1 and χ_2 have been defined in such a way that they provide the required viscous corrections to the leading terms of the singular inviscid solutions ϕ_1 and ϕ_2. The role played by these solutions is discussed further in § 9 where a more detailed derivation of χ_2 is also given. The remaining solutions are essentially of the usual viscous type and have been defined so that, as $\beta \to 0$, χ_3 and χ_4 tend to the corresponding solutions of the Orr–Sommerfeld equation.

6. The generalized Tietjens function $F(z, p)$

Before we derive the various approximations to the characteristic equation it will be useful to consider the generalized Tietjens function defined by

$$F(z,p) = \frac{A_1(\xi_1, p)}{\xi_1 A_1'(\xi_1, p)} \quad \text{with} \quad \xi_1 = z e^{-\frac{5}{6}\pi i} \quad \text{and } p \text{ real.} \tag{6·1}$$

In the case of neutral stability, z is real and positive but in some parts of the present discussion we will allow z to be an unrestricted complex variable. The universal function $F(z,p)$ plays an important role in the present theory and in this section, therefore, we wish to discuss some of its properties and describe a simple method of computing it.

For integral values of p, $F(z,p)$ can be expressed in terms of the usual Tietjens function $F(z)$ and the adjoint Tietjens function $F^\dagger(z)$ which arises in connexion with the asymptotic theory of the adjoint Orr–Sommerfeld equation (Reid 1965). Thus, for example, we have

$$F(z, -1) = F^\dagger(z), \tag{6·2}$$

$$F(z, 0) = \frac{i}{z^3\{1-F(z)\}F^\dagger(z)} = \frac{i\mathscr{F}(z)}{z^3 F^\dagger(z)}, \tag{6·3}$$

$$F(z, 1) = F(z), \tag{6·4}$$

and

$$F(z, 2) = \frac{1}{2}\left\{1 + F^\dagger(z) - \frac{F^\dagger(z)}{F(z)}\right\}, \tag{6·5}$$

where $\mathscr{F}(z) = \{1-F(z)\}^{-1}$ is the modified Tietjens function.

From equation (A 6) we have the alternate form

$$F(z,p) = -\frac{A_1(\xi_1, p)}{\xi_1 A_1(\xi_1, p-1)} \tag{6·6}$$

9-2

and from the recursion formula (A7) we can then obtain the relation

$$pF(z,p) = 1 - \frac{\mathrm{i}}{z^3 F(z,p-1)\,F(z,p-2)} \tag{6·7}$$

between three contiguous generalized Tietjens functions. Since $F(z, \pm 1)$ have already been tabulated it is convenient to note the further relation

$$pF(z,p) = 1 + (p-1)\,F(z,p-3) - \frac{F(z,p-3)}{F(z,p-1)}. \tag{6·8}$$

On setting $p = 1$ in equation (6·7) and solving for $F(z, 0)$ we obtain equation (6·3); similarly, on setting $p = 2$ in equation (6·8) we immediately obtain eqution (6·5). These results also show that we need only consider $F(z,p)$ for values of p in the range $-1 \leqslant p \leqslant 2$ (say).

Since $F(z,p)$ has a simple pole at $z = 0$, it is convenient for some purposes to let

$$H(z,p) = zF(z,p). \tag{6·9}$$

It then follows from equation (A1) that $H(z,p)$ satisfies the second-order non-linear equation

$$HH'' - 1 + 3H' - 2H'^2 + \mathrm{e}^{-\frac{1}{2}\pi\mathrm{i}}(zH^2 - pH^3) = 0 \tag{6·10}$$

and the initial conditions

$$H(0,p) = -\mathrm{e}^{\frac{5}{6}\pi\mathrm{i}}\frac{\Gamma(\frac{2}{3}+\frac{1}{3}p)}{3^{\frac{1}{3}}\Gamma(1+\frac{1}{3}p)} \quad \text{and} \quad H'(0,p) = 1 - \frac{\{\Gamma(\frac{2}{3}+\frac{1}{3}p)\}^2}{\Gamma(1+\frac{1}{3}p)\,\Gamma(\frac{1}{3}+\frac{1}{3}p)}. \tag{6·11}$$

The leading term in the asymptotic expansion of $F(z,p)$ can, of course, be obtained from equation (A9) and this shows that $F(z,p)$ has a purely neutral expansion in the sector $\frac{1}{6}\pi < \arg z < \frac{3}{2}\pi$. Additional terms in this expansion can be obtained most easily by using equation (6·10) and in this way we obtain

$$F(z,p) = \mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}z^{-\frac{3}{2}} + \tfrac{1}{4}(3+2p)\,\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}}z^{-3} + \tfrac{1}{32}(59+72p+20p^2)\,\mathrm{e}^{\frac{3}{4}\pi\mathrm{i}}z^{-\frac{9}{2}} + O(|z|^{-6}). \tag{6·12}$$

The complete asymptotic expansion of $F(z,p)$ in the sector $-\frac{1}{2}\pi < \arg z < \frac{1}{6}\pi$ could, if needed, be obtained from the results given in the appendix but it would be of a much more complicated form. For the present purposes, however, it is sufficient to note that the expansion (6·12) remains valid, in the sense of Poincaré, in the larger sector $-\frac{1}{6}\pi < \arg z < \frac{11}{6}\pi$.

To compute $H(z,p)$, and hence $F(z,p)$, for real values of z it is convenient to rewrite equation (6·10) as a system of real first-order, equations. For this purpose we let

$$H = X + \mathrm{i}Y \quad \text{and} \quad H' = U + \mathrm{i}V \tag{6·13}$$

and obtain

$$\left.\begin{aligned} X' &= U, \\ Y' &= V, \\ U' &= \frac{X\{1-3U+2(U^2-V^2)\} - YV(3-4U)}{X^2+Y^2} - zY + 2pXY, \\ \text{and} \qquad V' &= \frac{-Y\{1-3U+2(U^2-V^2)\} - XV(3-4U)}{X^2+Y^2} + zX - p(X^2-Y^2). \end{aligned}\right\} \tag{6·14}$$

The required initial values then follow from (6·11) in the form

$$\left.\begin{aligned} X(0) &= \frac{\sqrt{3}}{2}\,\frac{\Gamma(\frac{2}{3}+\frac{1}{3}p)}{3^{\frac{1}{3}}\Gamma(1+\frac{1}{3}p)}, & Y(0) &= -\frac{1}{2}\,\frac{\Gamma(\frac{2}{3}+\frac{1}{3}p)}{3^{\frac{1}{3}}\Gamma(1+\frac{1}{3}p)}, \\ U(0) &= 1-\frac{\{\Gamma(\frac{2}{3}+\frac{1}{3}p)\}^2}{\Gamma(1+\frac{1}{3}p)\,\Gamma(\frac{1}{3}+\frac{1}{3}p)}, & V(0) &= 0. \end{aligned}\right\} \tag{6·15}$$

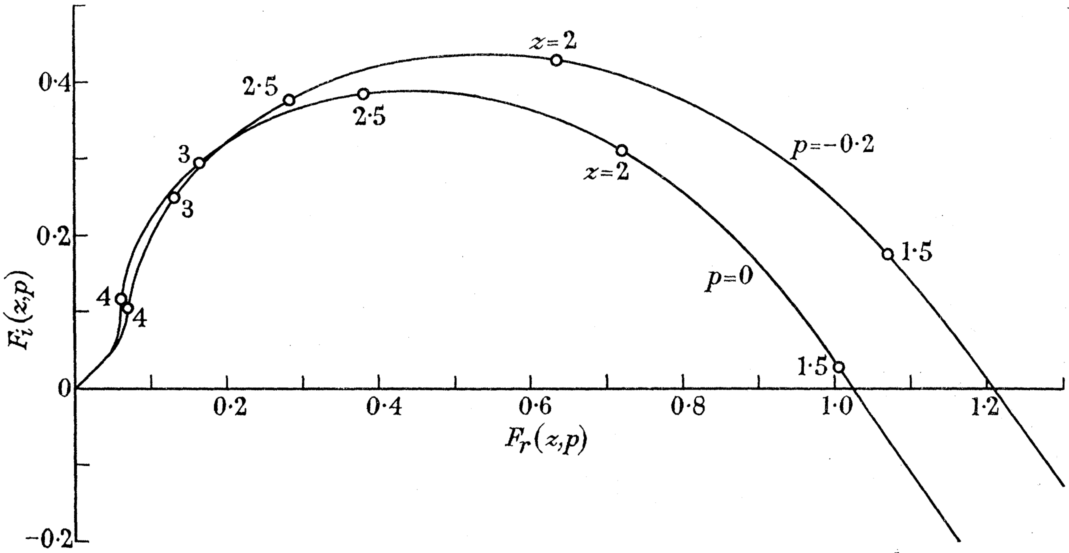

FIGURE 1. The generalized Tietjens function $F(z, p)$ for $p = 0$ and $-0·2$.

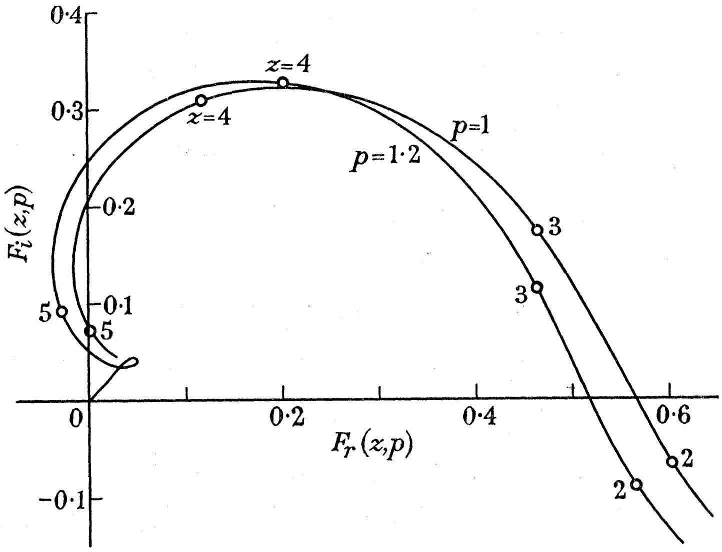

FIGURE 2. The generalized Tietjens function $F(z, p)$ for $p = 1$ and $1·2$. There is also a loop (not shown) for $p = 1$.

For fixed values of p this system of first-order equations can be integrated without difficulty by using a fourth-order single precision Runge–Kutta method with a step size of $\Delta z = 0·0125$. For comparison purposes $F(z,p)$ was also computed for $p = -1(1)\,2$ and $z = 0(0·1)\,10$ from existing tables of the modified Hankel functions of order one-third and their integrals (Harvard University Computation Laboratory 1945; Singh, Lumley & Betchov 1963), and complete agreement was obtained. Some results of these calculations are shown in figures 1 and 2 for typical values of p.

7. The characteristic equation using viscous solutions of the local turning point type

Having obtained approximations to six solutions of the governing equation, we can now use the properties of these solutions to derive simplified forms of the characteristic equation. The simplest approximation to the characteristic equation is obtained by ignoring the viscous corrections to ϕ_1 and ϕ_2 altogether and by using the local turning point approximations to the viscous solutions. The effect of ignoring the viscous corrections will be examined later in § 9. In using the local turning point approximations to the viscous solutions, however, we are effectively assuming that the values of c along the curves of neutral stability remain small, i.e. that the critical point remains close to the boundary $y = -1$. Since the value of c associated with the minimum critical Reynolds number does, in fact, increase from 0·27 to 0·78 as β increases from 0 to ∞, this is an approximation that may become questionable for sufficiently large values of β. Fortunately, however, this limitation can easily be overcome by using approximations to the viscous solutions of the composite type first suggested by Tollmien (1947). The derivation of these composite solutions will be given in § 10 where we will also examine their effect on the characteristic equation.

In the central part of the channel we expect that viscous effects will be negligible and consequently we must reject χ_4 and χ_6. Furthermore, since χ_3 and χ_5 are exponentially small at $y = 0$, they automatically satisfy the boundary conditions there with an exponentially small error. Thus, if we let

$$\Phi = A\phi_1 + \phi_2 \tag{7·1}$$

be the solution of the inviscid equation that satisfies the boundary condition $\Phi'(0) = 0$, then, since $U(y)$ is an even function of y, it automatically satisfies the other boundary conditions $\Phi'''(0) = \Phi^{\mathrm{v}}(0) = 0$.

Consider then an approximation to the solution of equation (2·32) of the form

$$\phi = \Phi + C_3\chi_3 + C_5\chi_5. \tag{7·2}$$

Differentiation of this result immediately gives

$$\phi' = \Phi' + C_3\epsilon^{-1}\chi_3' + C_5\epsilon^{-1}\chi_5' \tag{7·3}$$

but the evaluation of $L_4\phi$ from (7·2) and the satisfaction of the third boundary condition at $y = -1$ requires more careful consideration. If the operator L_4 is formally applied to Φ, we obtain

$$L_4\Phi = \epsilon^3(\mathrm{D}^2-\alpha^2)^2\,\Phi - \beta^2(U-c)^{-1}\,\Phi, \tag{7·4}$$

where we have used the fact that Φ satisfies equation (3·2). Thus, away from the critical point we have

$$L_4\Phi \to -\beta^2(U-c)^{-1}\,\Phi, \tag{7·5}$$

with an error $O(|\epsilon|^3)$. Near the critical point, however, the ratio of the viscous to the inviscid terms in (7·4) is $O(|\xi|^{-3})$ and the approximation (7·5) then remains valid provided $|\xi|^{-3} \ll 1$. Similarly, applying the operator L_4 to a viscous solution χ and using the fact that χ satisfies equation (5·5) we have

$$L_4\chi \to \epsilon^{-1}U_c'(1-p)\,\chi' \tag{7·6}$$

with an error $O(1)$ provided $|y-y_c| \ll 1$. On combining these results we obtain

$$L_4\phi \to -\frac{\beta^2}{U-c}\,\Phi + C_3\epsilon^{-1}U_c'(1-p_1)\,\chi_3' + C_5\epsilon^{-1}U_c'(1-p_2)\,\chi_5'. \tag{7·7}$$

The characteristic equation then follows from equations (7·2), (7·3), and (7·7) in the form

$$\begin{vmatrix} \Phi(-1) & \chi_3(\xi_1) & \chi_5(\xi_1) \\ \Phi'(-1) & \epsilon^{-1}\chi_3'(\xi_1) & \epsilon^{-1}\chi_5'(\xi_1) \\ (\beta^2/c)\,\Phi(-1) & \epsilon^{-1}U_c'(1-p_1)\,\chi_3'(\xi_1) & \epsilon^{-1}U_c'(1-p_2)\,\chi_5'(\xi_1) \end{vmatrix} = 0, \tag{7·8}$$

where ξ_1 denotes the value of ξ at $y=-1$, i.e. $\xi_1 = -(1+y_c)/\epsilon$. On expansion and simplification we obtain

$$\Delta_1(\alpha, c, z; \beta) \equiv \frac{1}{1+y_c}(p_1-p_2)+\frac{\Phi'(-1)}{\Phi(-1)}\{p_1F(z,p_1)-p_2F(z,p_2)\} + \frac{U_c'}{c}p_1p_2\{F(z,p_1)-F(z,p_2)\} = 0, \tag{7·9}$$

where $z = \xi_1\,e^{\frac{5}{6}\pi i}$ as usual and $F(z,p)$ is the generalized Tietjens function discussed in § 6. Note that if we let $\beta \to 0$ so that $p_1 \to 1$ and $p_2 \to 0$ and interpret ϕ_2 as the logarithmic solution of Rayleigh's equation then equation (7·9) reduces to the usual form

$$\frac{1}{1+y_c}\frac{\Phi(-1)}{\Phi'(-1)} = -F(z). \tag{7·10}$$

Even in this simple approximation the characteristic equation (7·9), unlike its limiting form (7·10), is not 'separable' and, as a result, direct methods of solution, such as the usual graphical method, cannot be used.

For fixed values of β, the zeros of Δ_1 define a curve of neutral stability in the (α, R)-plane. Along such a neutral curve z increases from about 2 to ∞ and, as usual, we use z as the second fixed parameter. For given values of β and z ($\bar{z}$ say) the zeros of Δ_1 are found by Newton's method and this requires many evaluations of Δ_1 at different values of α and c. Thus, it is necessary to consider carefully the manner in which we evaluate Δ_1. Since the p_i, $i = 1, 2$, depend on c, $F(\bar{z}, p_i)$ must be computed for each evaluation of Δ_1 that requires a new value of c. This procedure is very time consuming but can be avoided if we let $\eta = 2\beta/U_c'$ and consider the values of η rather than β to be fixed. The p_i are then functions only of η and hence we can calculate an entire neutral curve with only one computation of $F(z, p_i)$.

To find the zeros of Δ_1 which is a complex-valued non-linear function of the two real parameters α and c, we used the two-dimensional form of Newton's method

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \mathbf{J}^{-1}(\mathbf{f}(\mathbf{x}))\,\mathbf{f}(\mathbf{x}), \tag{7·11}$$

where $\mathbf{x} = (\alpha, c)$, $\mathbf{f} = (f, g) = (\mathscr{R}(\Delta_1), \mathscr{I}(\Delta_1))$, and $\mathbf{J}^{-1}$ is the inverse of the Jacobian (see, Mack 1965; Isaacson & Keller 1966). Equation (7·11) can be put into a more convenient form by writing it as a system of two linear equations for $\mathbf{x}_{n+1} = (\alpha_{n+1}, c_{n+1})$, i.e.

$$\left.\begin{aligned} (\alpha_{n+1}-\alpha_n)f_\alpha(\alpha_n, c_n) + (c_{n+1}-c_n)f_c(\alpha_n, c_n) + f(\alpha_n, c_n) &= 0, \\ \text{and} \quad (\alpha_{n+1}-\alpha_n)\,g_\alpha(\alpha_n, c_n) + (c_{n+1}-c_n)\,g_c(\alpha_n, c_n) + g(\alpha_n, c_n) &= 0. \end{aligned}\right\} \tag{7·12}$$

The partial derivatives f_α, f_c, g_α, and g_c are obtained approximately by computing $\Delta_1(\alpha_n, c_n)$, $\Delta_1(\alpha_n+\Delta\alpha, c_n)$ and $\Delta_1(\alpha_n, c_n+\Delta c)$, and then forming the appropriate ratios. This method is quadratically convergent if the initial guess, $\mathbf{x}_0$, is 'sufficiently close' to the root. For a given value of η and the first fixed value of z ($\bar{z}_1$ say), the initial guess, $\mathbf{x}_0$, can be found by plotting

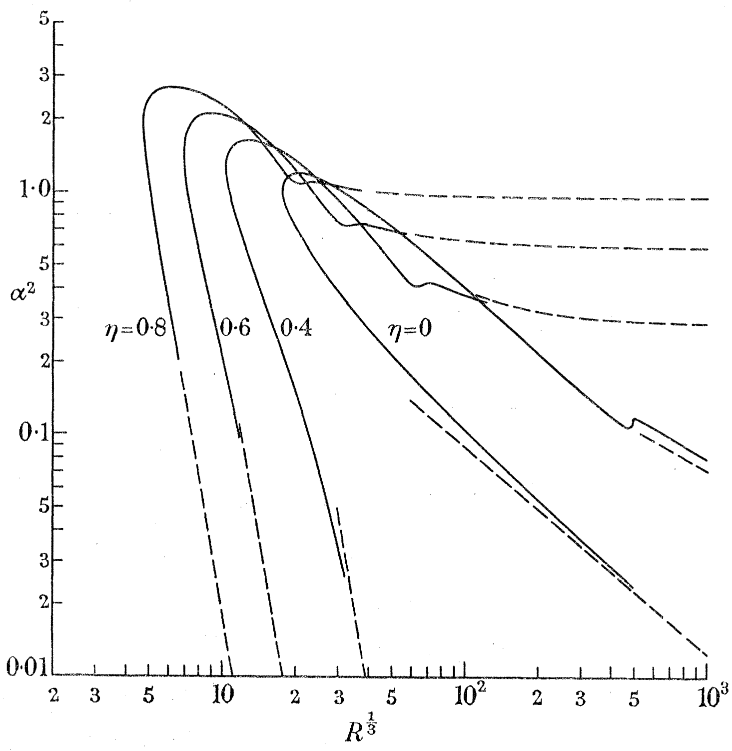

FIGURE 3. The curves of neutral stability for η = 0, 0·4, 0·6, and 0·8. The kinks along the upper branches of these curves are due to the loops in $F(z, p)$ shown in figure 2. ----, Asymptotes to the upper and lower branches.

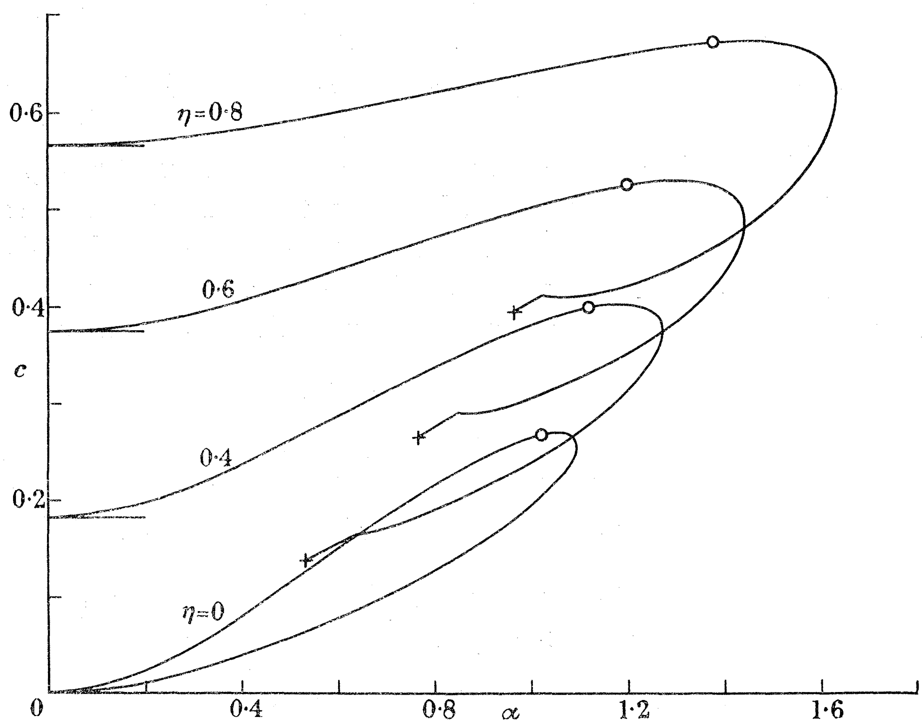

FIGURE 4. The relation between the wave-number α and the wave-speed c along the neutral curves. The points ○ correspond to the minimum critical Reynolds number. What appear to be corners in these curves are actually small loops (cf. Hughes & Reid 1965*a*, *b*).

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the absolute value of Δ_1 on a coarse network of points in the (α, c)-plane. One or two subsequent plots on a finer network can be used to improve $\mathbf{x}_0$. Once the first root is found, the initial guess for $\bar{z}_2 = \bar{z}_1 + \Delta z$ can simply be taken as the root already found for $\bar{z}_1$. With this procedure only five or six iterations using equations (7·12) were needed to obtain the roots to five significant figures.

Table 1. Results based on the characteristic equation (7·9)

η	p_1	p_2	β	z	α	c	R	$\sqrt{T}$
0·00	1·0000	0·0000	0·0000	3·043	1·022	0·2672	5397·1	0·0
0·01	1·0000	0·0000	0·0086	3·042	1·022	0·2673	5390·0	184·6
0·05	1·0006	−0·0006	0·0427	3·038	1·024	0·2695	5221·7	892·5
0·10	1·0025	−0·0025	0·0851	3·025	1·030	0·2765	4739·6	1612·5
0·15	1·0056	−0·0056	0·1266	3·005	1·040	0·2879	4061·1	2056·2
0·20	1·0099	−0·0099	0·1669	2·974	1·052	0·3033	3314·7	2213·6
0·25	1·0154	−0·0154	0·2058	2·937	1·068	0·3225	2603·7	2143·1
0·30	1·0220	−0·0220	0·2428	2·892	1·085	0·3450	1988·1	1930·9
0·35	1·0297	−0·0297	0·2778	2·840	1·102	0·3702	1489·0	1654·4
0·40	1·0385	−0·0385	0·3104	2·784	1·121	0·3979	1102·1	1368·5
0·45	1·0483	−0·0483	0·3405	2·722	1·140	0·4276	810·9	1104·3
0·50	1·0590	−0·0590	0·3678	2·659	1·160	0·4590	595·6	876·2
0·55	1·0706	−0·0706	0·3921	2·593	1·180	0·4917	438·3	687·4
0·60	1·0831	−0·0831	0·4132	2·530	1·203	0·5257	323·8	535·5
0·65	1·0963	−0·0963	0·4308	2·469	1·230	0·5607	240·8	415·0
0·70	1·1103	−0·1103	0·4445	2·415	1·266	0·5967	180·4	320·7
0·75	1·1250	−0·1250	0·4540	2·369	1·312	0·6336	136·2	247·3
0·80	1·1403	−0·1403	0·4583	2·337	1·380	0·6719	103·5	189·9
0·85	1·1562	−0·1562	0·4558	2·318	1·482	0·7124	78·58	143·3
0·895	1·1710	−0·1710	0·4373	2·342	1·715	0·7613	57·28	100·2

The results of our calculations based on the characteristic equation (7·9) are given in table 1. The results end at $\eta = 0{\cdot}895$ because of a severe distortion of the neutral curves which appears to be physically unrealistic. A selection of curves of neutral stability are shown in figure 3 and the corresponding behaviour of the wave-speed c is shown in figure 4. These results are based on fixed values of η, and we must assure ourselves that the minimum values of R are close to the values that would have been found had β been kept fixed. As shown in figure 4, c is slowly varying in the neighbourhood of the minimum values and, therefore, η is nearly proportional to β and we should get approximately the same minima. Test calculations have confirmed that to within graphical accuracy the minima are the same. They also showed, however, that the calculations with η fixed are about fifty times faster than the calculations with β fixed.

8. The asymptotes to the curves of neutral stability

To complete the discussion of the neutral curves, based on the approximation (7·9) to the characteristic equation, it is necessary to obtain their asymptotic behaviour as $R \to \infty$. Since the limiting behaviour of the upper and lower branches are quite different, the two limits must be treated separately. Along the upper branch we approach a purely inviscid limit but one that is necessarily singular. Along the lower branch, however, the critical point remains at a finite distance from the boundary $y = -1$ as $R \to \infty$ and viscous effects do not therefore become negligible.

The asymptotic behaviour of the upper branch of the neutral curve

Consider first the limiting inviscid solution, which we will denote by $\Phi_s(y)$. One of the results obtained by Miles (1961) in his study of the inviscid theory for stratified parallel flows is applicable to the present problem and it implies that Φ_s must be a multiple of *either* ϕ_1 *or* ϕ_2. Thus, within the framework of a purely inviscid theory, there is an indeterminacy in the limiting inviscid solution. From the present viscous theory, however, we find that as $R \to \infty$ along the upper branch $A \to 0$, $\alpha \to \alpha_s$, $c \to c_s$, and $z \to \infty$. The limiting inviscid solution must therefore be a multiple of ϕ_2. Thus, we simply let $\Phi_s(y) = \phi_2(y;\, \alpha_s, c_s)$, where the 'eigenvalues' α_s and c_s must be determined so that ϕ_2 satisfies the inviscid equation

$$(U-c_s)^2\,(\phi_2''-\alpha_s^2\phi_2)-(U-c_s)\,U''\phi_2-\tfrac{1}{4}\eta^2U_s'^2\phi_2 = 0 \tag{8·1}$$

and the boundary conditions

$$\phi_2'(0) = 0 \quad \text{and} \quad \phi_2(-1) = 0. \tag{8·2}$$

Some typical values of the parameters associated with this limiting inviscid solution are given in table 2.

TABLE 2. THE VALUES OF THE PARAMETERS ASSOCIATED WITH THE LIMITING INVISCID SOLUTION

η	α_s	c_s	y_s	$\Phi_s(0)$	$\Phi_s'(-1)\,e^{p_2\pi i}$
0·0	0·0000	0·0000	−1·0000	—	—
0·2	0·2719	0·0385	−0·9806	26·00	54·08
0·4	0·5317	0·1380	−0·9284	7·278	16·08
0·6	0·7666	0·2657	−0·8569	3·854	8·936
0·8	0·9653	0·3960	−0·7772	2·703	6·379

To determine the way in which this limit is approached it is necessary to obtain the behaviour of $\Phi(0)/\Phi'(0)$ to at least first order in $\alpha-\alpha_s$ and $c-c_s$ as $\alpha \to \alpha_s$ and $c \to c_s$. This can formally be done without difficulty by means of the expansion procedure that was previously developed for ordinary boundary layers with an inflexion point (Hughes & Reid 1965*b*). For this purpose it is convenient to consider a second solution, $\Psi_s(y)$ say, of the limiting inviscid equation (8·1). A natural choice for this second solution is simply $\Psi_s(y) = \phi_1(y;\alpha_s, c_s)$ and the Wronskian of these solutions is then $W(\Phi_s, \Psi_s) = p_1-p_2$.

Consider now an expansion of $\Phi(y)$ in powers of both $\alpha-\alpha_s$ and $c-c_s$ of the form

$$\Phi(y) = \Phi_s(y)+\Phi_1(y)\,(\alpha-\alpha_s)+\Phi_2(y)\,(c-c_s)+\ldots, \tag{8·3}$$

where $\Phi(y)$ satisfies equation (3·2). For fixed values of η, Φ_1 and Φ_2 must satisfy the inhomogeneous equations

$$\left.\begin{aligned} M\Phi_1 &= 2\alpha_s(U-c_s)^2\,\Phi_s \\ \text{and}\qquad M\Phi_2 &= \left\{U''-\eta^2+\tfrac{1}{2}\,\frac{\eta^2U_s'^2}{U-c_s}\right\}\Phi_s, \end{aligned}\right\} \tag{8·4}$$

where

$$M \equiv (U-c_s)^2\,(\mathrm{D}^2-\alpha_s^2)-(U-c_s)\,U''-\tfrac{1}{4}\eta^2U_s'^2. \tag{8·5}$$

The solutions of these equations that satisfy the boundary conditions $\Phi_1'(0) = \Phi_2'(0) = 0$ are

$$\Phi_1(y) = A_1\Phi_s-\frac{2\alpha_s}{p_1-p_2}\left\{\Psi_s\int_y^0\Phi_s^2\,\mathrm{d}y+\Phi_s\int_{y_s}^y\Phi_s\Psi_s\,\mathrm{d}y\right\} \tag{8·6}$$

and $$\Phi_2(y) = A_2\Phi_s - \frac{1}{p_1-p_2}\Psi_s\int_y^0\left\{U''-\eta^2+\frac{1}{2}\frac{\eta^2U_s'^2}{U-c_s}\right\}\frac{\Phi_s^2}{(U-c_s)^2}\mathrm{d}y$$
$$-\frac{1}{p_1-p_2}\Phi_s\int_{-1}^y\left\{U''-\eta^2+\frac{1}{2}\frac{\eta^2U_s'^2}{U-c_s}\right\}\frac{\Phi_s\Psi_s}{(U-c_s)^2}\mathrm{d}y. \quad (8\cdot7)$$

In these equations the paths of integration must, where necessary, be taken to lie below the critical point y_s. At $y = -1$, independently of the choice of A_1 and A_2, we have

$$\Phi_1(-1) = \frac{2\alpha_s}{\Phi_s'(-1)}\int_{-1}^0\Phi_s^2\,\mathrm{d}y \quad (8\cdot8)$$

and $$\Phi_2(-1) = \frac{1}{\Phi_s'(-1)}\int_{-1}^0\left\{U''-\eta^2+\frac{1}{2}\frac{\eta^2U_s'^2}{U-c_s}\right\}\frac{\Phi_s^2}{(U-c_s)^2}\mathrm{d}y, \quad (8\cdot9)$$

where we have used the Wronskian relation to express $\Psi_s(-1)$ in terms of $\Phi_s'(-1)$. In keeping with our usual normalization convention, the constants A_1 and A_2 could be fixed by the requirement that the coefficient of $(y-y_s)^{p_2}$ in the expansions of Φ_1 and Φ_2 vanish, i.e. that Φ_1 and Φ_2 be less singular than Φ_s. From this requirement it immediately follows that $A_1 = 0$ and that Φ_1 is then of order $(y-y_s)^{p_1}$ near y_s; the determination of A_2 is more complicated, however, and will be omitted since it is not needed for the present purposes.

These results can be used not only to obtain the asymptotic behaviour of the upper branch of the neutral curve as $R\to\infty$ but also to demonstrate the existence, near the neutral mode Φ_s, of a neighbouring unstable solution. For this latter purpose we consider the purely inviscid form of the characteristic equation (7·9) and immediately find that

$$c-c_s \to -\frac{\Phi_1(-1)\,\Phi_2^*(-1)}{|\Phi_2(-1)|^2}(\alpha-\alpha_s) \quad \text{as} \quad c\to c_s \quad \text{and} \quad \alpha\to\alpha_s. \quad (8\cdot10)$$

This result is substantially equivalent to Howard's formula (1963, equation (14)) for $c'(\alpha_s)$, which he obtained in the context of the inviscid theory for stratified shear flows. One important difference, of course, is that in the present analysis we have kept η rather than β fixed. On taking the imaginary part of equation (8·10) we obtain

$$c_i \to K_1(\alpha-\alpha_s), \quad \text{where} \quad K_1 = -\frac{\mathscr{I}\{\Phi_1(-1)\,\Phi_2^*(-1)\}}{|\Phi_2(-1)|^2}. \quad (8\cdot11)$$

The calculations which will be described below show that K_1 is negative and hence that $c_i \gtrless 0$ when $\alpha \lessgtr \alpha_s$.

To obtain the asymptotic behaviour of the upper branch of the neutral curve we must consider the asymptotic form of the characteristic equation (7·9) as $R\to\infty$ with $c_i = 0$. For this purpose it is convenient to let

$$a_1+\mathrm{i}b_1 = \Phi_1(-1)/\Phi_s'(-1) \quad \text{and} \quad a_2+\mathrm{i}b_2 = \Phi_2(-1)/\Phi_s'(-1). \quad (8\cdot12)$$

To first order in $\alpha-\alpha_s$ and $c-c_s$ we then have

$$-\frac{1}{1+y_s}\{(a_1+\mathrm{i}b_1)(\alpha-\alpha_s)+(a_2+\mathrm{i}b_2)(c-c_s)+\dots\}\sim\frac{\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}}{z^{\frac{3}{2}}}, \quad (8\cdot13)$$

where $$z = (\alpha_sRU_s')^{\frac{1}{3}}(1+y_s)\{1+O(\alpha-\alpha_s, c-c_s)\}. \quad (8\cdot14)$$

By eliminating z between the real and imaginary parts of equation (8·12), we obtain

$$c-c_s \to K_2(\alpha-\alpha_s), \quad \text{where} \quad K_2 = -\frac{a_1-b_1}{a_2-b_2}, \tag{8·15}$$

and the imaginary part itself then gives

$$R \to K_3(\alpha-\alpha_s)^{-2}, \quad \text{where} \quad K_3 = \frac{1}{2\alpha_s\,(1+y_s)\,U'_s}\left(\frac{a_2-b_2}{a_2 b_1-a_1 b_2}\right)^2. \tag{8·16}$$

It is perhaps worth noting that the dependence on $\alpha-\alpha_s$ given by equations (8·11), (8·15), and (8·16) is the same as for ordinary parallel flows with an inflexion point.

The principal difficulty in actually determining the coefficients K_1, K_2, and K_3 lies in the calculation of $\Phi_1(-1)$ and $\Phi_2(-1)$ from equations (8·8) and (8·9). For the present purposes it is sufficient to assume that $p_2 > -\frac{1}{2}$. The singularity in the integrand of equation (8·8) is then integrable and $\Phi_1(-1)$ can be evaluated numerically without difficulty. In the case of $\Phi_2(-1)$, however, the singular behaviour of the integrand of equation (8·9) is more serious. To deal with this situation we let

$$I_1 = \int_{-1}^{0} (U-c_s)^{-2}\,\Phi_s^2\,\mathrm{d}y \quad \text{and} \quad I_2 = \int_{-1}^{0} (U-c_s)^{-3}\,\Phi_s^2\,\mathrm{d}y. \tag{8·17}$$

Since $U'' = -2$ in the present problem these are the only integrals that need be considered. We now wish to transform these integrals so that only real integrals with integrable singularities need be computed. This can easily be done by first letting

$$f_1(y) = \left\{\frac{y-y_s}{U-c_s}\,\frac{\Phi_s(y)}{(y-y_s)^{p_2}}\right\}^2 \quad \text{and} \quad f_2(y) = \frac{y-y_s}{U-c_s}\,f_1(y) \tag{8·18}$$

with $f_1(y_s) = f_2(y_s) = 1$ so that they are both analytic at y_s. Repeated integration by parts then gives

$$I_1 = -\frac{y_s\,\Phi_s^2(0)}{p_2(1-c_s)^2} - \frac{1}{2p_2(1-2p_2)\,U_s'^2}\left\{\mathrm{e}^{-2p_2\pi \mathrm{i}}\int_{-1}^{y_s} + \int_{y_s}^{0}\right\}|y-y_s|^{2p_2} f_1''(y)\,\mathrm{d}y \tag{8·19}$$

and

$$I_2 = -\frac{3y_s\,\phi_s^2(0)}{2p_2(1-c_s)^3} - \frac{1}{4p_2(1-p_2)\,(1-2p_2)}\left\{\left[2\alpha_s^2 - \frac{U''(0)}{1-c_s} + \tfrac{1}{2}\eta^2\,\frac{U_s'^2}{(1-c_s)^2}\right]\frac{y_s^3\,\Phi_s^2(0)}{(1-c_s)^3}\right.$$
$$\left. + \frac{2(1+y_s)^3\,\Phi_s'^2(-1)}{c_s^3} + \frac{1}{U_s'^3}\left[\mathrm{e}^{-2p_2\pi\mathrm{i}}\int_{-1}^{y_s} + \int_{y_s}^{0}\right]|y-y_s|^{2p_2} g'''(y)\,\mathrm{d}y\right\}. \tag{8·20}$$

In deriving these results we have assumed only that $U(0) = 1$, $U'(0) = 0$, and $U(-1) = 0$. The numerical evaluation of these improper integrals is somewhat lengthy and we have therefore made detailed calculations for only three values of η. These results are given in table 3.

TABLE 3. THE VALUES OF THE PARAMETERS ASSOCIATED WITH THE ASYMPTOTE TO THE UPPER BRANCH OF THE NEUTRAL CURVE

η	$\Phi_1(-1)$	I_1	I_2	$\Phi_2(-1)$
0·4	1·90 − 0·231i	78·3 + 31·0i	50·5 + 113i	−9·85 − 1·05i
0·6	1·41 − 0·369i	24·2 + 20·2i	+6·09 + 39·0i	−6·61 − 1·38i
0·8	1·26 − 0·538i	10·6 + 16·9i	−0·175 + 23·9i	−5·74 − 1·83i

η	K_1	K_2	K_3
0·4	−0·0436	0·231	10080
0·6	−0·0961	0·279	560
0·8	−0·148	0·285	96·0

The asymptotic behaviour of the lower branch of the neutral curve

As $R \to \infty$ along the lower branch we find that $A \to A_\ell$, $\alpha \to 0$, $c \to c_\ell$, and $z \to z_\ell$. In this limit $\Phi \to \Phi_\ell$, where $\Phi_\ell(y)$ is the solution of the equation

$$(U-c_\ell)^2\,\Phi_\ell'' - (U-c_\ell)\,U''\Phi_\ell - \tfrac{1}{4}\eta^2 U_\ell'^2\Phi_\ell = 0 \tag{8·21}$$

that satisfies the boundary condition $\Phi_\ell'(0) = 0$. In this equation $U_\ell' = U'(y_\ell)$, where y_ℓ is the value of y for which $U - c_\ell = 0$. The value of c_ℓ is not known *a priori* but must be determined simultaneously with z_l from the limiting form of the characteristic equation

$$\Delta_1(0, c_\ell, z_\ell; \beta) = 0. \tag{8·22}$$

In this limit we also have

$$z_\ell \to (\alpha R U_\ell')^{\frac{1}{3}}\,(1+y_\ell) \tag{8·23}$$

from which it follows that

$$R \to K_\ell \alpha^{-1}, \quad \text{where} \quad K_\ell = \frac{1}{U_\ell'}\left(\frac{z_\ell}{1+y_\ell}\right)^3. \tag{8·24}$$

The values of the parameters associated with this limit are given in table 4.

TABLE 4. THE VALUES OF THE PARAMETERS ASSOCIATED WITH THE ASYMPTOTE TO THE LOWER BRANCH OF THE NEUTRAL CURVE

η	c_ℓ	y_ℓ	z_ℓ	A_ℓ	K_ℓ
0·0	0·0000	−1·0000	2·297	—	—
0·2	0·0467	−0·9764	2·244	51·54	438500
0·4	0·1829	−0·9039	2·122	14·25	5970
0·6	0·3740	−0·7912	1·997	7·632	552·9
0·8	0·5663	−0·6586	1·905	5·711	131·8

The approach to this limit can also be obtained by considering an expansion of the form

$$\Phi(y) = \Phi_\ell(y) + \Phi_3(y)\,\alpha^2 + \Phi_4(y)\,(c-c_\ell) + \dots, \tag{8·25}$$

where $\Phi_\ell(-1) \neq 0$. The subsequent analysis, however, is even more complicated than that given for the upper branch and it will therefore be omitted. The main conclusion from such an analysis is simply that $c - c_s \to \text{constant} \times \alpha^2$ as $\alpha \to 0$ (cf. figure 4).

9. THE EFFECT OF THE VISCOUS CORRECTION TO ϕ_2 ON THE CHARACTERISTIC EQUATION

Because of the singular character of the inviscid solutions, particularly ϕ_2, it would appear desirable to consider the effect of the viscous corrections on the characteristic equation. In the case of ϕ_1, the singularity in ϕ_1' and the inviscid form of $L_4\phi_1$ is as weak as $(y-y_c)^{p_1-1}$ and it would appear therefore that we need not consider the corrected form of ϕ_1. In the case of ϕ_2, however, which is singular like $(y-y_c)^{p_2}$ with p_2 negative we know, by analogy with the theory of the adjoint Orr–Sommerfeld equation (Hughes & Reid 1965*a*), that if p_2 were to become as negative as -1 then it would be essential to use the corrected form of ϕ_2.

To obtain the first viscous correction to ϕ_2 we must consider a viscous solution, χ_2 say, that is asymptotic (for $|\xi| \gg 1$) to ϕ_2 (for $|y-y_c| \ll 1$) in sectors of the complex y-plane that include the boundary points. From the results given in the appendix it immediately follows that the required viscous solution is given by

$$\chi_2(\xi) = \epsilon^{p_2} Q_3(\xi, p_2) \tag{9·1}$$

and that it is unique since Q_3 is the only solution of equation (A 1) for $p < 0$ that is neutral in the sector $S_1 \cup S_2$. If we now let ψ_2 denote a corrected form of ϕ_2, then with one viscous correction we have

$$\psi_2(y) = \phi_2(y) - (y-y_c)^{p_2} + \chi_2(\xi), \tag{9·2}$$

where the remaining singularity in ψ_2 is of the order of $(y-y_c)^{p_2+1}$.

Accordingly we now let

$$\Psi = A\phi_1 + \psi_2, \tag{9·3}$$

where A has the same meaning as in equation (7·1). In this approximation we are effectively assuming that $\psi_2 \to \phi_2$ as $y \to 0$ and hence that the coefficient A is determined by the condition $\Phi'(0) = 0$. The question then arises as to the meaning of L_4 operating on the term $(y-y_c)^{p_2}$ which appears in the definition of ψ_2. If we again invoke the principle that the boundary conditions need be satisfied only to the same order of approximation as the solutions themselves then we need retain only those parts of L_4 that are directly responsible for the leading term in ϕ_2 and in this way we obtain

$$L_4(y-y_c)^{p_2} \to -U_c'(y-y_c)\,\mathrm{D}^2(y-y_c)^{p_2} = U_c' p_2(1-p_2)\,(y-y_c)^{p_2-1}. \tag{9·4}$$

Thus we have

$$L_4\Psi \to -\frac{\beta^2}{U-c}\Phi - U_c' p_2(1-p_2)\,(y-y_c)^{p_2-1} + \epsilon^{p_2-1}U_c'(1-p_2)\,Q_3'(\xi, p_2), \tag{9·5}$$

where the remaining singularity is then as weak as $(y-y_c)^{p_2}$ and this would appear to be entirely acceptable provided $-1 < p_2 < 0$.

In this approximation the characteristic equation becomes

$$\begin{vmatrix} (1+y_c)^{-p_2}\mathrm{e}^{p_2\pi\mathrm{i}}\Phi(-1) - 1 + \xi_1^{-p_2}Q_3(\xi_1, p_2) & \chi_3(\xi_1) & \chi_5(\xi_1) \\ (1+y_c)^{p_1}\mathrm{e}^{-p_1\pi\mathrm{i}}\Phi'(-1) - p_2 + \xi_1^{p_1}Q_3'(\xi_1, p_2) & \xi_1\chi_3'(\xi_1) & \xi_1\chi_5'(\xi_1) \\ 2p_1p_2(1+y_c)^{p_1-1}\mathrm{e}^{-p_1\pi\mathrm{i}}y_c(1-y_c)^{-1}\Phi(-1) & & \\ \qquad -p_1p_2 + p_1\xi_1^{p_1}Q_3'(\xi_1, p_2) & p_2\xi_1\chi_3'(\xi_1) & p_1\xi_1\chi_5'(\xi_1) \end{vmatrix} = 0. \tag{9·6}$$

On expansion and simplification this becomes

$$\begin{aligned} &\Delta_2(\alpha, c, z; \beta) \\ &\quad \equiv \{(1+y_c)^{-p_2}\mathrm{e}^{p_2\pi\mathrm{i}}\Phi(-1) - 1 + \xi_1^{-p_2}Q_3(\xi_1, p_2)\}\,(p_1 - p_2) \\ &\qquad - \{(1+y_c)^{p_1}\mathrm{e}^{-p_1\pi\mathrm{i}}\Phi'(-1) - p_2 + \xi_1^{p_1}Q_3'(\xi_1, p_1)\}\{p_1F(z, p_1) - p_2F(z, p_2)\} \\ &\qquad + \{2p_1p_2(1+y_c)^{p_1-1}\mathrm{e}^{-p_1\pi\mathrm{i}}(1-y_c)^{-1}\Phi(-1) - p_1p_2 + p_1\xi_1^{p_1}Q_3'(\xi_1, p_2)\}\{F(z, p_1) \\ &\qquad - F(z, p_2)\} = 0. \end{aligned} \tag{9·7}$$

When $Q_3(\xi_1, p_2)$ can be represented approximately by the leading term in its asymptotic expansion then this result reduces, apart from a constant factor, to equation (7·9).

The viscous correction $Q_3(\xi, p)$ was computed by numerical integration of the differential equation

$$Q_3''' - \xi Q_3' + pQ_3 = 0 \tag{9·8}$$

with the initial conditions (cf. equation (A 25))

$$\left.\begin{aligned} Q_3(0, p) &= -\frac{3^{-\frac{1}{3}p}\Gamma(1-\frac{1}{3}p)}{p\Gamma(-p)}\,\mathrm{e}^{-\frac{1}{3}p\pi\mathrm{i}}, \\ Q_3'(0, p) &= +\frac{3^{\frac{1}{3}(1-p)}\,\Gamma(\frac{4}{3}-\frac{1}{3}p)}{(p-1)\,\Gamma(-p)}\,\mathrm{e}^{\frac{1}{3}(1-p)\pi\mathrm{i}}, \\ \text{and}\qquad Q_3''(0, p) &= -\frac{3^{\frac{1}{3}(2-p)}\,\Gamma(\frac{5}{3}-\frac{1}{3}p)}{(p-2)\,\Gamma(-p)}\,\mathrm{e}^{\frac{1}{3}(2-p)\pi\mathrm{i}}. \end{aligned}\right\} \tag{9·9}$$

A comparison of the results obtained in this way with existing tables (Reid 1965) of $Q_3(\xi, -1)$ confirmed the adequacy of this approach for values of z in the neighbourhood of the minimum critical Reynolds number. For larger values of z (of the order of 5) this method becomes inaccurate and greater care would be needed if one wished to compute the whole neutral curve in this approximation. An example of the behaviour of Q_3 is shown in figure 5.

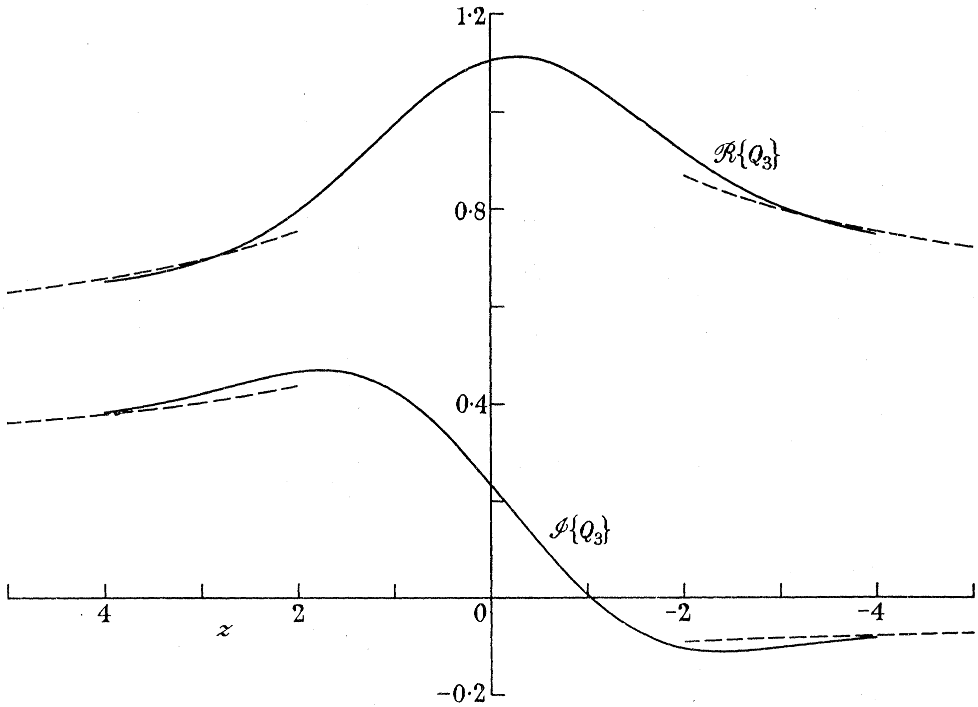

FIGURE 5. The viscous correction $Q_3(\xi_1, p_2)$ for $p_2 = -0{\cdot}2$.

The results of our calculations based on the characteristic equation (9·7) are given in table 5. A comparison of these results with those given in table 1 show that the effect of including the viscous correction to ϕ_2 in the characteristic equation is not large. This conclusion is perhaps not altogether surprising in view of the very mild singularity in ϕ_2 over the range of η that we have considered.

For values of η greater than about 0·5, however, the assumption that $\psi_2 \rightarrow \phi_2$ as $y \rightarrow 0$ begins to fail and it then becomes necessary to reconsider the boundary conditions at the centre of the channel. For this purpose we assume, as is usually done, that away from the critical point Φ does not change its order of magnitude on differentiation. The same is then true of Ψ since Q_3, unlike the other viscous solutions, is a slowly varying function. On this hypothesis we find that A must be determined by the condition $\Psi'(0) = 0$, i.e.

$$A\phi_1'(0) + \phi_2'(0) - (-y_c)^{p_2-1}\{p_2 - \xi_0^{1-p_2} Q_3'(\xi_0, p_2)\} = 0, \tag{9·10}$$

where

$$\xi_0 = -y_c/\epsilon = z_0 e^{\frac{1}{6}\pi i} \quad \text{(say)}. \tag{9·11}$$

With A determined in this way the boundary conditions $\phi'''(0) = \phi^{\mathrm{v}}(0) = 0$ are not satisfied exactly by Ψ. But the error thereby introduced into the characteristic equation is of order

$|\epsilon|^2$ compared with the terms retained in equation (9·7) and can therefore be neglected. In this approximation A also depends on z_0 but since z_0 is related to c and z through the equation

$$z_0 = \frac{|y_c|}{1-|y_c|}\,z \tag{9·12}$$

Table 5. Results based on the characteristic equation (9·7)

η	β	z	α	c	R	$\surd T$
0·00	0·0000	3·043	1·022	0·2672	5397·1	0·0
0·01	0·0086	3·042	1·022	0·2673	5389·8	184·5
0·05	0·0427	3·038	1·024	0·2696	5216·6	891·7
0·10	0·0850	3·024	1·030	0·2767	4721·1	1606·1
0·15	0·1265	3·000	1·040	0·2883	4025·4	2037·7
0·20	0·1668	2·967	1·053	0·3041	3262·6	2177·4
0·25	0·2056	2·925	1·068	0·3238	2538·9	2087·9
0·30	0·2424	2·874	1·085	0·3470	1915·6	1857·7
0·35	0·2771	2·815	1·103	0·3732	1413·3	1566·5
0·40	0·3092	2·750	1·122	0·4023	1026·9	1270·4
0·45	0·3386	2·681	1·142	0·4338	739·0	1000·8
0·50	0·3647	2·611	1·165	0·4678	529·2	772·2
0·55	0·3874	2·543	1·192	0·5040	378·7	586·6
0·60	0·4059	2·482	1·229	0·5423	271·7	441·2
0·65	0·4200	2·429	1·276	0·5825	196·1	329·4
0·70	0·4289	2·385	1·340	0·6245	142·4	244·3
0·75	0·4315	2·356	1·434	0·6690	103·5	178·6
0·80	0·4226	2·350	1·609	0·7209	72·72	123·0
0·815	0·4073	2·370	1·782	0·7503	59·70	97·25
			With the centre correction (9·10)			
0·55	0·3874	2·543	1·192	0·5040	378·7	586·6
0·60	0·4059	2·480	1·226	0·5423	271·8	441·2
0·65	0·4192	2·423	1·274	0·5841	193·3	324·1
0·70	0·4255	2·376	1·357	0·6305	134·8	229·5
0·75	0·4252	2·355	1·497	0·6787	94·75	161·1
0·80	0·4144	2·369	1·737	0·7316	65·96	109·34
0·808	0·4060	2·385	1·850	0·7475	59·21	96·15

there is no further difficulty in the subsequent solution of the characteristic equation. From the results given in table 5 it can be seen that the effect of this 'centre correction' is completely negligible for $0 \leqslant \eta \leqslant 0{\cdot}55$ and even for larger values of η the effect remains small.

10. The characteristic equation for composite solutions of the Tollmien type

Because of the relatively large values of c that occur in this problem it is desirable to consider briefly the composite solutions of the Tollmien type which include both the W.K.B. and local turning point approximations as limiting cases. For this purpose we first introduce the Langer variable

$$\zeta = (\mathrm{i}\alpha R)^{\frac{1}{3}}\left[\frac{3}{2}\int_{y_c}^{y}(U-c)^{\frac{1}{2}}\,\mathrm{d}y\right]^{\frac{2}{3}} \tag{10·1}$$

and note that

$$\tfrac{2}{3}\zeta^{\frac{3}{2}} = (\alpha R)^{\frac{1}{2}}Q(y), \tag{10·2}$$

where $Q(y)$ is given by equation (4·15). A comparison of the W.K.B. solutions (4·13) and (4·14) with the local turning point solutions (5·8) then suggests that we can construct composite solutions of the form

$$\left.\begin{aligned}\phi_3(y) &= G_1(y)\,(\epsilon\zeta)^{-q_1}A_1(\zeta,p_1),\\ \phi_4(y) &= G_1(y)\,(\epsilon\zeta)^{-q_1}A_2(\zeta,p_1),\\ \phi_5(y) &= G_2(y)\,(\epsilon\zeta)^{-q_2}A_1(\zeta,p_2),\\ \text{and}\qquad \phi_6(y) &= G_2(y)\,(\epsilon\zeta)^{-q_2}A_2(\zeta,p_2),\end{aligned}\right\} \tag{10·3}$$

where G_1 and G_2 are given by equations (4·11). For small values of $|y-y_c|$, $\zeta\to\xi$ and these composite solutions then reduce to the local turning point solutions (5·8). For large values of $|\zeta|$ they reduce to the W.K.B. solutions (4·13) and (4·14) but only in certain sectors of the complex ζ-plane. For ϕ_3 and ϕ_5 this reduction is possible (in the strict sense) only in the sector $|\arg\zeta| < \frac{2}{3}\pi$. This sector does not include the boundary point $y=-1$ and this fact explains the poor results that are obtained when the W.K.B. approximations are used for stability calculation (see, for example, Reid 1965). For ϕ_4 and ϕ_6 this reduction is possible (again in the strict sense) in the sector $-\frac{4}{3}\pi < \arg\zeta < 0$ and this sector does not contain the other boundary point $y=0$.

When approximations of this type are differentiated, only the terms that arise from differentiation of the rapidly varying part of the solutions can consistently be retained. Thus, for example, we have

$$\phi_3'(y) = G_1(y)\,(\epsilon\zeta)^{-q_1}\zeta' A_1'(\zeta,p_1), \tag{10·4}$$

where $\zeta\zeta'^2 = \mathrm{i}\alpha R(U-c)$. Similarly, in applying the operator L_4 to a composite solution we can approximate L_4 by its 'truncated' form, i.e.

$$L_4 \to (\mathrm{i}\alpha R)^{-1}\mathrm{D}^4-(U-c)\,\mathrm{D}^2, \tag{10·5}$$

and thus obtain

$$L_4\phi_3(y) \to G_1(y)\,(\epsilon\zeta)^{-q_1}(\mathrm{i}\alpha R)^{-1}\zeta'^4(1-p_1)\,A_1'(\zeta,p_1). \tag{10·6}$$

On using the viscous solutions of composite type together with the uncorrected inviscid solutions we obtain the characteristic equation in the form

$$\begin{vmatrix} \Phi(-1) & A_1(\zeta_1,p_1) & A_1(\zeta_1,p_2) \\ \Phi'(-1) & \zeta_1' A_1'(\zeta_1,p_1) & \zeta_1' A_1'(\zeta_1,p_2) \\ (\beta^2/c)\,\Phi(-1) & (\mathrm{i}\alpha R)^{-1}\zeta_1'^4(1-p_1)\,A_1'(\zeta_1,p_1) & (\mathrm{i}\alpha R)^{-1}\zeta_1'^4(1-p_2)\,A_1'(\zeta_1,p_2) \end{vmatrix} = 0 \tag{10·7}$$

where ζ_1 and ζ_1' denote the values of ζ and ζ' at $y=-1$. To conform to our previous notation as closely as possible we now let

$$\zeta_1 = \hat{z}\,\mathrm{e}^{-\frac{5}{6}\pi\mathrm{i}}, \quad \text{where} \quad \hat{z} = (\alpha R)^{\frac{1}{3}}\left[\frac{3}{2}\int_{-1}^{y_c}|U-c|^{\frac{1}{2}}\,\mathrm{d}y\right]^{\frac{2}{3}}. \tag{10·8}$$

When $U(y) = 1-y^2$ we have

$$\hat{z} = (\alpha R)^{\frac{1}{3}}\{\tfrac{3}{4}\mu(c)\}^{\frac{2}{3}} \quad \text{and} \quad \frac{\zeta_1'}{\zeta_1} = -\frac{4}{3}\frac{\surd c}{\mu(c)}, \tag{10·9}$$

where

$$\mu(c) = \surd c-(1-c)\ln(1+\surd c)+\tfrac{1}{2}(1-c)\ln(1-c). \tag{10·10}$$

On expansion and simplification the characteristic equation then becomes

$$\begin{aligned}\Delta_3(\alpha,c,\hat{z};\beta) \equiv \frac{4}{3}\frac{\surd c}{\mu(c)}(p_1-p_2)+\frac{\Phi'(-1)}{\Phi(-1)}\{p_1F(\hat{z},p_1)-p_2F(\hat{z},p_2)\}&\\ +3p_1p_2c^{-\frac{5}{2}}(1-c)\,\mu(c)\,\{F(\hat{z},p_1)-F(\hat{z},p_2)\} = 0.&\end{aligned} \tag{10·11}$$

For small values of c this reduces, as it should, to equation (7·9).

The effect of using the Langer variable in the viscous solutions can be partially illustrated by comparing the dependence of z and $\hat{z}$ on c as shown in figure 6. For this purpose we have let

$$z = (\alpha R)^{\frac{1}{3}} f(c) \quad \text{and} \quad \hat{z} = (\alpha R)^{\frac{1}{3}} \hat{f}(c), \tag{10·12}$$

where $$f(c) = 2^{\frac{1}{3}}(1-c)^{\frac{1}{6}}\{1-(1-c)^{\frac{1}{2}}\} \quad \text{and} \quad \hat{f}(c) = \{\tfrac{3}{4}\mu(c)\}^{\frac{2}{3}}. \tag{10·13}$$

The results of our calculations based on the characteristic equation (10·11) are given in table 6.

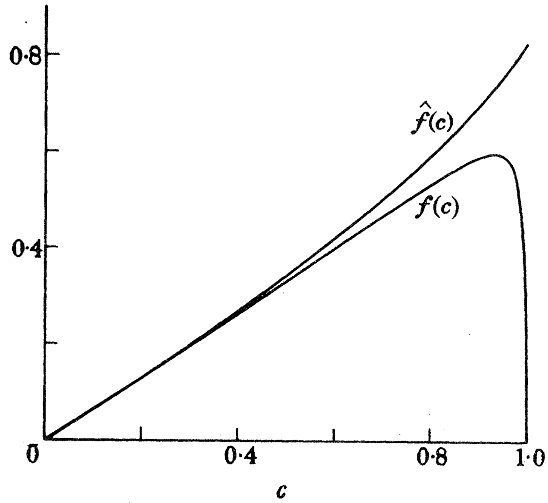

FIGURE 6. The behaviour of $f(c)$ and $\hat{f}(c)$.

TABLE 6. RESULTS BASED ON THE CHARACTERISTIC EQUATION (10·11)

η	β	z	α	c	R	$\sqrt{T}$
0·00	0·0000	3·058	1·010	0·2607	5697·3	0·0
0·01	0·0086	3·057	1·010	0·2608	5689·8	195·7
0·05	0·0429	3·054	1·012	0·2630	5512·6	946·6
0·10	0·0855	3·040	1·017	0·2696	5004·9	1710·9
0·15	0·1272	3·020	1·027	0·2805	4290·3	2183·5
0·20	0·1679	2·990	1·038	0·2954	3503·9	2353·0
0·25	0·2071	2·952	1·051	0·3136	2754·3	2281·4
0·30	0·2446	2·907	1·066	0·3351	2104·7	2059·5
0·35	0·2802	2·853	1·080	0·3589	1577·4	1767·4
0·40	0·3137	2·795	1·095	0·3851	1167·7	1464·9
0·45	0·3448	2·729	1·109	0·4129	858·3	1184·0
0·50	0·3736	2·655	1·119	0·4418	628·8	939·6
0·55	0·3997	2·577	1·127	0·4719	460·2	735·6
0·60	0·4230	2·496	1·135	0·5029	336·9	569·9
0·65	0·4433	2·417	1·144	0·5350	247·0	438·0
0·70	0·4602	2·339	1·155	0·5678	181·6	334·2
0·75	0·4734	2·270	1·173	0·6016	134·1	253·9
0·80	0·4827	2·209	1·200	0·6360	99·69	192·4
0·85	0·4877	2·159	1·241	0·6708	74·72	145·8
0·90	0·4885	2·116	1·294	0·7054	56·49	110·4
0·95	0·4842	2·090	1·373	0·7402	43·04	83·36
1·00	0·4743	2·068	1·468	0·7750	32·90	62·41
1·05	0·4535	2·079	1·654	0·8135	24·69	44·79
1·065	0·4388	2·109	1·801	0·8302	21·88	38·40

It is perhaps worth while to note briefly how the first viscous correction to ϕ_2 can also be introduced. For this purpose it is convenient to define an unstretched Langer variable

$$\eta = \left[\frac{3}{2}\int_{y_c}^{y}\left(\frac{U-c}{U_c'}\right)^{\frac{1}{2}} \mathrm{d}y\right]^{\frac{2}{3}} \tag{10·14}$$

so that $\zeta = \eta/\epsilon$. If we think of Φ as having been expressed in terms of η then it is clear that the corrected inviscid solution must be of the form

$$\Psi = \Phi - \eta^{p_2} + \epsilon^{p_2} Q_3(\zeta, p_2). \tag{10·15}$$

To evaluate $L_4 \Psi$ we must consider further the meaning of L_4 operating on η^{p_2}. As in § 9 this can involve only the highest derivative in the inviscid part of L_4 when expressed in terms of η, i.e.

$$L_4 \eta^{p_2} \to -U_c' \eta'^4 \eta \frac{d^2}{d\eta^2} \eta^{p_2} = U_c' p_2 (1-p_2) \eta'^4 \eta^{p_2-1}. \tag{10·16}$$

Thus we have (cf. equation (9·5))

$$L_4 \Psi \to -\beta^2 (U-c)^{-1} \Phi - U_c' p_2 (1-p_2) \eta'^4 \eta^{p_2-1} + \epsilon^{p_2-1} U_c' (1-p_2) \eta'^4 Q_3'(\zeta, p_2). \tag{10·17}$$

We have not, however, made any calculations based on this corrected form of the inviscid solution since they would not be expected to differ appreciably from those based on equation (10·11).

11. Discussion

One of the principal results of the present paper is the stability boundary in the $(R, \surd T)$-plane which is shown in figure 7 together with the theoretical results of Krueger & DiPrima (1964) and Datta (1965). Our results based on the uncorrected characteristic equation (7·9) have been omitted from this figure since they lie between the two curves shown. Although one would have expected the parameter β to be a monotonic function of R, it was found (see tables 1, 5 and 6) that β actually reached a maximum, the value of which varied somewhat from one form of the characteristic equation to another as indicated in figures 7, 8, and 9. Thereafter β decreased slightly until a point was reached where the structure of the neutral curves changed quite dramatically. Since the values of R at which β reaches its maxima are of the order of 100, it is likely that our asymptotic approximations are beginning to become unreliable beyond these points.

The perturbation theories of Chandrasekhar (1962) and Datta (1965) were derived on the assumption that R is small. Nevertheless, the agreement between Datta's formula and the present results extends to far larger values of R (of the order of 500) than one would have expected and this suggests that his result may be asymptotic in character. This agreement is even more surprising when it is recalled that in the perturbation theories the wave-number α is assumed to remain constant with the value 1·56.

For values of R up to about 15, Krueger & DiPrima (1964) have made a detailed comparison between their theoretical results and the experimental results of Snyder (1962) and Schwarz, Springett & Donnelly (1964) and found good agreement on the whole. For larger values of R, the only existing experimental results would appear to be those obtained by Kaye & Elgar (1957) and Williamson (1964). These two sets of data are in reasonable agreement for values of R up to about 300 but for larger values of R they are in substantial disagreement. The agreement with the theoretical results extends only to about $R = 75$ and at this point there is the suggestion of a 'break' (i.e. a discontinuity in the slope) of the experimental stability boundary. Williamson's data also show a large *increase* in the values of α (from 1·56 to about 3·75) as R increases from 0 to about 350 and this trend is certainly at variance with the theoretical results shown in figure 9.

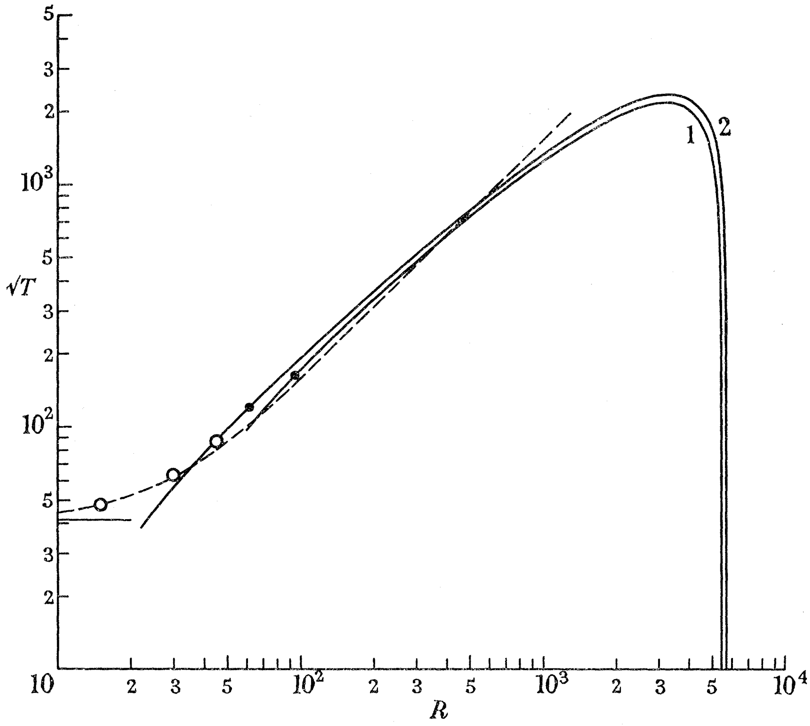

FIGURE 7. The stability boundary in the $(R, \sqrt{T})$-plane. Curves 1 and 2 are based on the characteristic equations (9·7) and (10·11) respectively. – – – –, Datta's result $\sqrt{T} = (1708 + 2{\cdot}35R^2)^{\frac{1}{2}}$; ○, from Krueger & DiPrima (1964); ●, where β reaches a maximum.

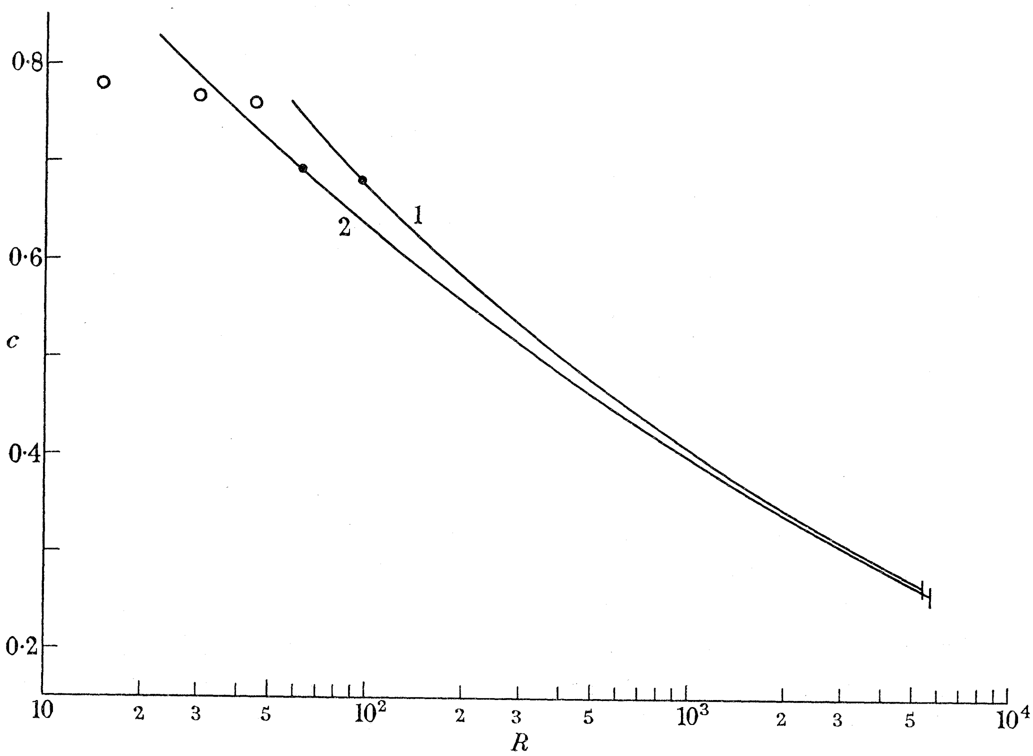

FIGURE 8. The variation of c with R. Curve 1 is based on the characteristic equations (7·9) and (9·7); curve 2 is based on equation (10·11). ○, From Krueger & DiPrima (1964); ●, where β reaches a maximum.

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In spite of the obvious limitations of the present theory, however, the asymptotic techniques which have been developed would appear to be applicable to a wide class of problems including, for example, the stability of thermally stratified plane Poiseuille flow (Gage & Reid 1968). The present results could also be used as starting values for a direct numerical attack on the problem as suggested recently by Reynolds & Potter (1967).

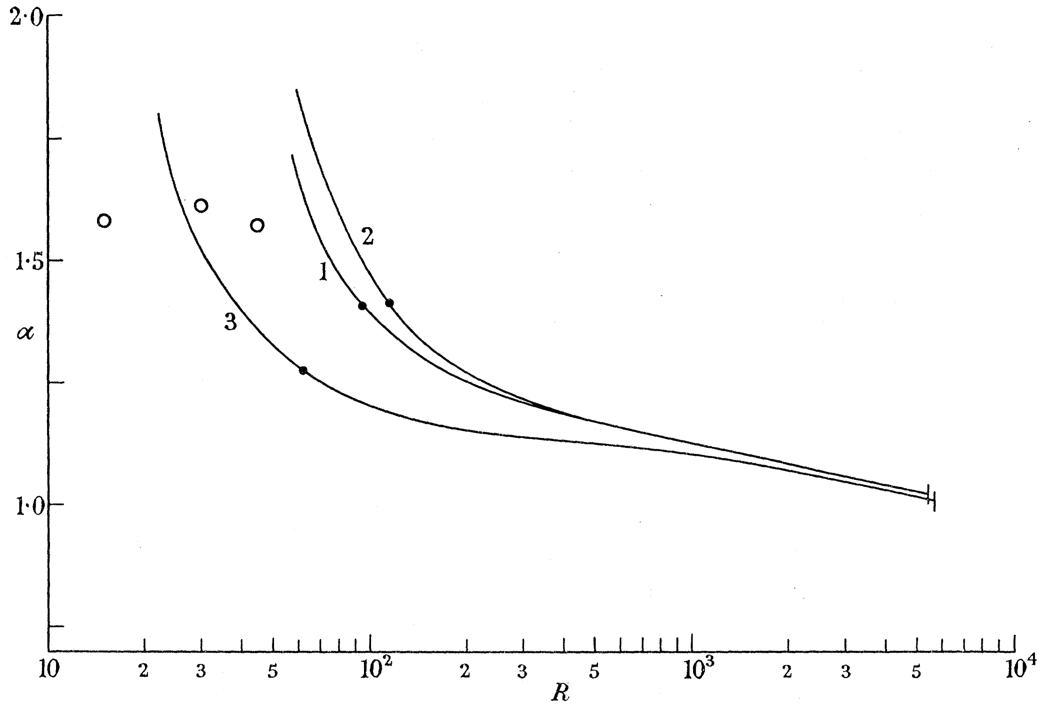

FIGURE 9. The variation of α with R. Curves 1, 2, and 3 are based on the characteristic equations (7·9), (9·7), and (10·11) respectively. ○, From Krueger & DiPrima (1964); ●, where β reaches a maximum.

Because of the limited agreement between the theoretical and experimental results it seems clear that other possible modes of instability must also be investigated. These would include extending the linear theory to allow for non-axisymmetric disturbances or, as Snyder has suggested to us, the development of a non-linear theory for subcritical instabilities. Either of these generalizations would involve severe theoretical difficulties and it may be hoped, therefore, that further experimental work will provide some guidance in choosing between these alternatives.

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APPENDIX. THE SOLUTIONS OF THE EQUATION $(\mathrm{AD}+p)\,\chi = 0$

We have seen in §5 that the solutions of the comparison equation (5·3) can all be obtained from the solutions of the simpler third-order equation

$$(\mathrm{AD}+p)\,\chi = 0 \qquad \text{(A 1)}$$

by assigning appropriate values to p. In this appendix, therefore, we wish to define certain standard solutions of this equation that are particularly well suited for the present purposes and to discuss some of their properties.

Solutions of equation (A1) can easily be obtained by the method of Laplace integrals in the form

$$\int_C t^{-p-1}\,\mathrm{e}^{(\xi t-\frac{1}{3}t^3)}\,\mathrm{d}t, \tag{A2}$$

where the path of integration C must be chosen so that

$$[t^{-p}\,\mathrm{e}^{(\xi t-\frac{1}{3}t^3)}]_C = 0. \tag{A3}$$

The integrand in the representation (A2) will, in general, be multiple-valued and it is convenient, therefore, to introduce a cut into the t-plane running from the origin to infinity along the positive real axis. For all values of p we can then choose three paths (C_1, C_2, and C_3) that run from infinity to infinity as shown in figure 10. The solutions associated with these paths have the property that they are subdominant in the sectors S_1, S_2, and S_3 of figure 11 respectively. When $p = -1$ they are closely related to the Airy functions defined by Olver (1954) and when $p = 1$ they are closely related to the usual approximations to the viscous solutions of the Orr–Sommerfeld equation.

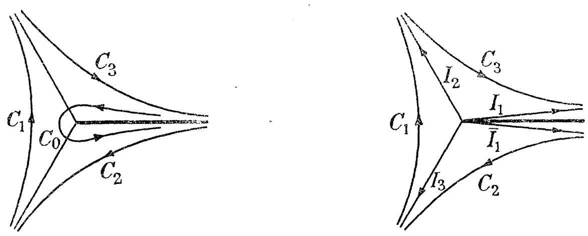

FIGURE 10. The paths of integration in the t-plane for $p \geqslant 0$ (left) and $p < 0$ (right).

FIGURE 11. The anti-Stokes lines (left) and the Stokes lines (right) in the ξ-plane.

If $p \geqslant 0$ it is also convenient to consider the solution associated with the path C_0 shown in figure 10. This solution has the property that in the sector T_1 of figure 11 its asymptotic expansion is purely neutral, i.e. contains no subdominant terms. From this solution we can derive two others that are purely neutral in T_2 and T_3 respectively. The solution that is purely neutral in T_3 provides the viscous correction to the leading term of ϕ_1 and thus determines the domain of validity of ϕ_1. If $p < 0$ we can also choose four paths (I_1, I_2, I_3, and $\bar{I}_1$) that run from the origin to infinity as shown in figure 10. The solutions associated with these paths have asymptotic expansions that are purely neutral in the sectors T_1, T_2, T_3, and T_1 respectively. The solution associated with the path I_3 provides the viscous correction to the leading term of ϕ_2 and thus determines the domain of validity of ϕ_2.

The solutions $A_k(\xi,p)$

Consider then the solution associated with the path C_1. In defining this solution we shall require, first, that it reduce to the usual Airy function Ai (ξ) when $p=-1$ and, secondly, that it be real (for all values of p) when ξ is real. Thus we define a standard solution $A_1(\xi,p)$ by the relation

$$A_1(\xi,p) = \frac{1}{2\pi \mathrm{i}}\,\mathrm{e}^{(p+1)\pi \mathrm{i}}\int_{C_1} t^{-p-1}\,\mathrm{e}^{(\xi t-\frac{1}{3}t^3)}\,\mathrm{d}t. \tag{A 4}$$

The solutions associated with the paths C_2 and C_3 can then be expressed in terms of $A_1(\xi,p)$ by the relations

$$\left.\begin{aligned} A_2(\xi,p) &= \mathrm{e}^{-\frac{2}{3}p\pi \mathrm{i}}\,A_1(\xi\,\mathrm{e}^{\frac{2}{3}\pi \mathrm{i}},p) \\ \text{and}\qquad A_3(\xi,p) &= \mathrm{e}^{\frac{2}{3}p\pi \mathrm{i}}\,A_1(\xi\,\mathrm{e}^{-\frac{2}{3}\pi \mathrm{i}},p).\end{aligned}\right\} \tag{A 5}$$

The derivatives of these solutions satisfy the relation

$$A_k^{(n)}(\xi,p) = (-1)^n\,A_k(\xi,p-n) \tag{A 6}$$

and from equation (A 1) we then have the recursion formula

$$A_k(\xi,p-3)-\xi A_k(\xi,p-1)-pA_k(\xi,p) = 0. \tag{A 7}$$

Accordingly it is sufficient to consider the solutions for values of p in the range $-1 \leqslant p \leqslant 2$ (say); this includes, however, the relevant ranges of p_1 and p_2 for the stability problem.

For integral values of p, $A_1(\xi,p)$ can be expressed in terms of the Airy function Ai(ξ) together with its first derivative and first integral. Thus, for example, we have

$$A_1(\xi,-1) = \mathrm{Ai}(\xi),$$

$$A_1(\xi,0) = -\int_{\infty_1}^{\xi} \mathrm{Ai}\,(t)\,\mathrm{d}t,$$

$$A_1(\xi,1) = \xi\int_{\infty_1}^{\xi} \mathrm{Ai}\,(t)\,\mathrm{d}t-\mathrm{Ai}'(\xi),$$

and

$$A_1(\xi,2) = -\frac{1}{2}\left\{\xi^2\int_{\infty_1}^{\xi} \mathrm{Ai}\,(t)\,\mathrm{d}t-\xi\mathrm{Ai}'(\xi)-\mathrm{Ai}\,(\xi)\right\},$$

where the lower limit of integration ∞_1 denotes a path of integration that tends to infinity in the sector S_1 of figure 11. For the numerical calculation of $A_1(\xi,p)$ and other related functions it is useful to note the initial value

$$A_1(0,p) = \frac{1}{3^{1+\frac{1}{3}p}\,\Gamma(1+\frac{1}{3}p)}. \tag{A 8}$$

The leading term in the asymptotic expansion of $A_1(\xi,p)$ can easily be obtained by the usual saddle point method and is given by

$$A_1(\xi,p) \sim \tfrac{1}{2}\pi^{-\frac{1}{2}}\xi^{-(2p+3)/4}\exp\left(-\tfrac{2}{3}\xi^{\frac{3}{2}}\right). \tag{A 9}$$

This result is valid in the sense of Poincaré in the sector $|\arg\xi| < \pi$ but, by analogy with the Airy functions, we may expect it to be valid in the stricter sense of Olver only in the smaller sector $|\arg\xi| < \frac{2}{3}\pi$. In this more restricted sense we see that A_1 (ξ,p) is subdominant in the sector $|\arg\xi| < \frac{1}{3}\pi$ and purely dominant in the sectors $\frac{1}{3}\pi < |\arg\xi| < \frac{2}{3}\pi$. The descending series associated with (A 9) could be obtained without difficulty by the method of steepest descents but it is not needed for the present purposes. From equations (A 5) and (A 9) it then follows that $A_k(\xi,p)$ is subdominant in S_k $(k = 1,2,3)$.

The complete asymptotic expansions of $A_k(\xi,p)$ in T_k are more complicated since, in general, they contain not only dominant and subdominant terms but also a neutral term. This is, of course, simply a consequence of the fact that we are dealing with a third-order differential equation. The required expansions can be obtained most easily by the use of exact connexion formulas and for this purpose we must now make a distinction between positive and negative values of p.

The solutions $B_k(\xi,p)$

For $p \geqslant 0$ we consider first the solution associated with the path C_0 which, for later convenience, is defined by the relation

$$B_1(\xi,p) = \frac{1}{2\pi \mathrm{i}}\Gamma(p+1)\int_{C_0} t^{-p-1}\,\mathrm{e}^{(\xi t-\frac{1}{3}t^3)}\,\mathrm{d}t. \tag{A 10}$$

It is also convenient to define two additional solutions of this type by the relations (cf. equations (A 5))

$$\left.\begin{aligned} B_2(\xi,p) &= \mathrm{e}^{-\frac{2}{3}p\pi\mathrm{i}}B_1(\xi\,\mathrm{e}^{\frac{2}{3}\pi\mathrm{i}},p)\\ \text{and}\qquad B_3(\xi,p) &= \mathrm{e}^{\frac{2}{3}p\pi\mathrm{i}}B_1(\xi\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}},p).\end{aligned}\right\} \tag{A 11}$$

The derivatives of these solutions satisfy the relation

$$B_k^{(n)}(\xi,p) = \frac{\Gamma(p+1)}{\Gamma(p+1-n)}B_1(\xi,p-n) \tag{A 12}$$

provided $p-n$ is not equal to a negative integer, and from equation (A 1) we then have the recursion formula

$$(p-1)(p-2)\,B_k(\xi,p-3)-\xi B_k(\xi,p-1)+B_k(\xi,p) = 0, \tag{A 13}$$

in which we must require $p \geqslant 3$. Thus it is sufficient to consider these solutions for values of p in the range $0 \leqslant p \leqslant 3$.

For integral values of p, the integrand in equation (A 10) is single-valued with a pole of order $p+1$ at $t = 0$ and the residue theorem then gives

$$B_1(\xi,0) = 1, \quad B_1(\xi,1) = \xi, \quad B_1(\xi,2) = \xi^2, \ldots. \tag{A 14}$$

We may also note the initial value

$$B_1(0,p) = -\frac{1}{p(p-3)\,\pi}3^{1-\frac{1}{3}p}\,\Gamma(p+1)\,\Gamma(2-\tfrac{1}{3}p)\,\mathrm{e}^{-p\pi\mathrm{i}}\sin p\pi, \tag{A 15}$$

which is valid for $0 \leqslant p \leqslant 3$; for $p > 3$ the values of $B_1(0,p)$ can be obtained by using this result together with the recursion formula (A 13).

The asymptotic expansion of $B_1(\xi,p)$ can easily be obtained by the method of integration by parts and the use of Hankel's contour integral representation of the gamma function, with the result

$$B_1(\xi,p) = \xi^p\{1-\tfrac{1}{3}p(p-1)\,(p-2)\,\xi^{-3}+O(|\xi|^{-6})\}. \tag{A 16}$$

This result is valid in the sense of Poincaré in the sector $-\frac{5}{3}\pi < \arg\xi < -\frac{1}{3}\pi$ but it is complete in the sense of Olver only in the smaller sector $-\frac{4}{3}\pi < \arg\xi < -\frac{2}{3}\pi$. From equations (A 11) and (A 16) it then follows that $B_k(\xi,p)$ is purely neutral in $T_k (k = 1,2,3)$.

Furthermore, for integral values of p the series (A16) terminates and we then recover the exact solutions (A14).

It is also of some interest to note the relation between (A16) and the result obtained by Koppel (1964), who used the saddle-point method to obtain the leading term in the asymptotic expansion of an integral equivalent to (A10). In the present notation, his result can be written in the form

$$B_1(\xi,p) \sim (2\pi)^{-\frac{1}{2}} e^{p+1}(p+1)^{-p-\frac{1}{2}}\, \Gamma(p+1)\, \xi^p, \tag{A17}$$

which does not reduce to the exact solutions (A14) for integral values of p. For large values of p, however, Stirling's formula shows that the coefficient of p in this expression does tend to unity.

The six solutions A_k and B_k are not, of course, linearly independent but must be related by three exact connexion formulae. From figure 10 it immediately follows that

$$A_1+A_2+A_3 = \frac{e^{p\pi i}}{\Gamma(p+1)} B_1. \tag{A18}$$

On replacing ξ in this equation by $\xi e^{\pm\frac{2}{3}\pi i}$ and using equations (A5) and (A11) we obtain the two additional connexion formulae

$$A_1+A_2+e^{-2p\pi i}A_3 = \frac{e^{p\pi i}}{\Gamma(p+1)} B_2 \quad \text{and} \quad A_1+e^{2p\pi i}A_2+A_3 = \frac{e^{p\pi i}}{\Gamma(p+1)} B_3. \tag{A19}$$

By using the expansions (A9) and (A16), the definitions (A5) and (A11), and the three connexion formulae, we can now obtain the complete asymptotic expansions of all of the solutions throughout the entire ξ-plane for $p \geqslant 0$. Consider, for example, the solution B_3 (ξ, p) which provides the required viscous correction to the leading term of ϕ_1. Its complete asymptotic expansion is easily found to be

$$B_3(\xi,p) = \xi^p\{1-\tfrac{1}{3}p(p-1)(p-2)\,\xi^{-3}+O(|\xi|^{-6})\} + \begin{cases} +2i\sin p\pi\, \Gamma(p+1)\, A_2(\xi,p) & (\xi \in T_1), \\ -2i\sin p\pi\, \Gamma(p+1)\, A_1(\xi,p) & (\xi \in T_2), \\ 0 & (\xi \in T_3), \end{cases} \tag{A20}$$

where $-\frac{4}{3}\pi < \arg\xi < \frac{2}{3}\pi$.

The solutions $Q_k(\xi,p)$

For $p < 0$ we must now consider the solutions associated with the paths I_k and $\bar{I}_1$ shown in figure 10. For this purpose we first define the solution

$$Q_1(\xi,p) = \frac{e^{p\pi i}}{\Gamma(-p)} \int_{I_1} t^{-p-1}\, e^{(\xi t-\frac{1}{3}t^3)}\, dt. \tag{A21}$$

The solutions associated with the other paths can then be defined by the relations

$$\left.\begin{aligned} Q_2(\xi,p) &= e^{-\frac{2}{3}p\pi i}\, Q_1(\xi e^{\frac{2}{3}\pi i},p), \\ Q_3(\xi,p) &= e^{-\frac{4}{3}p\pi i}\, Q_1(\xi e^{\frac{4}{3}\pi i},p), \\ \text{and} \qquad \bar{Q}_1(\xi,p) &= e^{-2p\pi i}\, Q_1(\xi e^{2\pi i},p). \end{aligned}\right\} \tag{A22}$$

The derivatives of these solutions satisfy the relation

$$Q_k^{(n)}(\xi,p) = (-1)^n \frac{\Gamma(n-p)}{\Gamma(-p)}\, Q_k(\xi,p-n) \tag{A23}$$

and from equation (A 1) we then have the recursion formula

$$(p-1)(p-2)\,Q_k(\xi,p-3)-\xi Q_k(\xi,p-1)+Q_k(\xi,p)=0 \tag{A 24}$$

which is of exactly the same form as equation (A 13).

For integral values of p no particular simplification is possible but it may be noted that $Q_3(\xi,-1)=Q_3(\xi)$, where $Q_3(\xi)$ is the viscous function defined and tabulated (for $\arg\xi=-\frac{5}{6}\pi$) by Reid (1965) in connexion with the asymptotic theory of the adjoint Orr–Sommerfeld equation. We also note the initial value

$$Q_1(0,p)=-\frac{e^{p\pi i}}{p3^{\frac{1}{3}p}}\,\frac{\Gamma(1-\frac{1}{3}p)}{\Gamma(-p)}. \tag{A 25}$$

The method of integration by parts can again be used to obtain the asymptotic expansion

$$Q_1(\xi,p)=\xi^p\{1-\tfrac{1}{3}p(p-1)(p-2)\,\xi^{-3}+O(|\xi|^{-6})\}, \tag{A 26}$$

which, though identical in form with equation (A 16), is valid (in the complete sense) in the sector $\frac{2}{3}\pi<\arg\xi<\frac{4}{3}\pi$. From equations (A 22) it then follows that Q_2, Q_3, and $\overline{Q}_1$ are purely neutral in the sectors T_2, T_3, and T_1 respectively.

Thus, for $p<0$ we have seven solutions which must be related by four connexion formulae. These follow, by inspection, from figure 10 in the form

$$\left.\begin{aligned} 2\pi i(A_1+A_2+A_3)+\Gamma(-p)\,(Q_1-\overline{Q}_1)&=0,\\ 2\pi i A_1+\Gamma(-p)\,(Q_2-Q_3)&=0,\\ 2\pi i A_2+\Gamma(-p)\,(Q_3-\overline{Q}_1)&=0,\\ \text{and}\qquad 2\pi i A_3+\Gamma(-p)\,(Q_1-Q_2)&=0.\end{aligned}\right\} \tag{A 27}$$

By using the expansions (A 9) and (A 26), the definitions (A 5) and (A 22), and these four connexion formulae, we can obtain the complete asymptotic expansions of all of the solutions throughout the entire ξ-plane for $p<0$. Consider, for example, the solution $Q_3(\xi,p)$ which provides the required viscous corrections to the leading term of ϕ_2. Its complete asymptotic expansion is found to be

$$Q_3(\xi,p)=\xi^p\{1-\tfrac{1}{3}p(1-p)(2-p)\,\xi^{-3}+O(|\xi|^{-6})\}+\left\{\begin{array}{ll} -\dfrac{2\pi i}{\Gamma(-p)}A_2(\xi,p) & (\xi\in T_1),\\ +\dfrac{2\pi i}{\Gamma(-p)}A_1(\xi,p) & (\xi\in T_2),\\ 0 & (\xi\in T_3),\end{array}\right\} \tag{A 28}$$

where $-\frac{4}{3}\pi<\arg\xi<\frac{2}{3}\pi$.

The solutions for half integral values of p

For half integral values of p it is not difficult to show that all of the solutions of equation (A 1) can be expressed in terms of products of Airy functions and their derivatives with argument $2^{-\frac{2}{3}}\xi=x$ (say). Thus, for example, we find

$$\left.\begin{aligned} A_1(\xi,-\tfrac{1}{2})&=2^{\frac{2}{3}}\pi^{\frac{1}{2}}\mathrm{Ai}^2(x),\\ A_1(\xi,+\tfrac{1}{2})&=2^{\frac{4}{3}}\pi^{\frac{1}{2}}\{\mathrm{Ai}'^2(x)-x\,\mathrm{Ai}^2(x)\},\\ Q_3(\xi,-\tfrac{1}{2})&=2^{\frac{5}{3}}\pi\,e^{\frac{1}{6}\pi i}\,\mathrm{Ai}(x)\,\mathrm{Ai}(x\,e^{\frac{2}{3}\pi i})\\ \text{and}\qquad B_3(\xi,+\tfrac{1}{2})&=2^{\frac{4}{3}}\pi\,e^{\frac{1}{6}\pi i}\{x\mathrm{Ai}(x)\,\mathrm{Ai}(x\,e^{\frac{2}{3}\pi i})\\ &\qquad -e^{\frac{2}{3}\pi i}\,\mathrm{Ai}'(x)\,\mathrm{Ai}'(x\,e^{\frac{2}{3}\pi i})\}.\end{aligned}\right\} \tag{A 29}$$

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